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Why Isn't There an Inverse Quotient Rule?

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Introduction

When learning Calculus, students are taught how to "undo the Chain Rule" through u-Substitution, and may have been introduced to Integration by Parts as a way to "undo the Product Rule". I will explain how a curious student could develop an integration technique to "undo the Quotient Rule". I will draw from my own experience teaching Calculus 1 and 2 to offer some commentary on whether an Inverse Quotient Rule is a good technique for a Calculus student to have in their toolbox!

I'm sure that none of this will be novel. I am writing this paper primarily to encourage interested Calculus students to think through how they might have discovered this rule themselves, and partly to have a set of exercises they can work on. If you are a Calculus student reading this paper, I encourage you to try to answer the questions I pose before I answer them myself, and to work through some of the exercises at the end of the paper.

This paper has two primary jumping-off points: the first is James Stewart's *Essential Calculus, Early Transcendentals, Second Edition* [5], which is currently the standard Calculus textbook at the University of Pittsburgh. The second is a video by the YouTube channel blackpenredpen (a.k.a. Steve Chow). His solution to one of Stewart's exercises [3] inspired me to think through this rule.

1. The Basic Integration Techniques

Let's remind ourselves of the integration techniques we know so far:

Theorem 1 (u-Substitution)**.** *If we can write an integral in the form* $\int f(u(x))u'(x)dx$ *for some functions* f *and* u*, where* u *is differentiable, then we may write*

$$\int f(u(x))u'(x)dx = \int f(u)du.$$

Proof. This strategy follows directly from the Chain Rule. If we write $\int f(u)du = F(u)$, where F is an anti-derivative of f, then the derivative of $F(u(x))$ with respect to x is

$F'(u(x))u'(x) = f(u(x))u'(x)$, which is exactly the integral we started with. See Chapter 5.5 of Stewart's textbook [5] for more details. □

Theorem 2 (Integration by Parts)**.** *If we can write an integral in the form* $\int f(x)g'(x)dx$ *for some differentiable functions* f *and* g*, then we may write*

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx.$$

Proof. This strategy follows from integrating both sides of the Product Rule:

$$\begin{aligned}(f(x)g(x))' &= f(x)g'(x) + f'(x)g(x)\\ \int (f(x)g(x))'dx &= \int f(x)g'(x)dx + \int f'(x)g(x)dx\\ f(x)g(x) &= \int f(x)g'(x)dx + \int f'(x)g(x)dx\\ \int f(x)g'(x)dx &= f(x)g(x) - \int f'(x)g(x)dx.\end{aligned}$$

This notation is sometimes changed so it can be written more cleanly:

$$\int fg' = fg - \int f'g.$$

You may have also seen this rule written in terms of functions u and v. I've switched the letters so it looks completely distinct from u-Substitution. See Chapter 6.1 of Stewart's textbook [5] for more details. □

Besides these, there are three more integration techniques worth mentioning for now:

- Trigonometric Integrals: These integrals often use trig identities,[1] so the strategies to solve them tend to be distinct from other integrals. See Chapter 6.2 of Stewart's textbook [5] for more details.

- Trigonometric Substitution: A variation of u-Substitution that introduces trig functions into an integral. Doing this often allows us to take advantage of their convenient identities. See Chapter 6.2 of Stewart's textbook [5] for more details.

- Partial Fraction Decomposition: An algebraic strategy that lets us turn certain complicated fractions into easier fractions. This is primarily useful when the numerator and denominator are both polynomials. See Chapter 6.3 of Stewart's textbook [5] for more details.

Stewart devotes an extra section to discuss Integration using Lookup Tables, which encourages us to rely on previous results, but that's it as far as integration techniques go.

[1] For example, $\sin^2(\theta) + \cos^2(\theta) = 1$ for all θ.

So at this point, knowing that integrals "undo" derivatives, we can list the differentiation rules we've managed to invert:

Differentiation	Power Rule	Product Rule	Chain Rule	**Quotient Rule**	Trig & Inv. Trig	Exp. & Log
"Undone" by Integrals?	Yes	Yes	Yes	**No**	Yes	Yes

Why did we not learn a rule for "undoing the Quotient Rule?" Taking inspiration from how we "undid the Product Rule," we can come up with one ourselves. See if you can do it yourself first:

$$\begin{aligned}\left(\frac{p}{q}\right)' &= \frac{qp' - pq'}{q^2} \\ \int \left(\frac{p}{q}\right)' &= \int \frac{qp' - pq'}{q^2} \\ \frac{p}{q} &= \int \frac{qp'}{q^2} - \int \frac{pq'}{q^2} \\ \frac{p}{q} &= \int \frac{p'}{q} - \int \frac{pq'}{q^2} \\ \boldsymbol{\int \frac{pq'}{q^2}} &\boldsymbol{= -\frac{p}{q} + \int \frac{p'}{q}}.\end{aligned}$$

So we have the following Theorem:

Theorem 3 (The Inverse Quotient Rule, Version 1). *If we can write an integral in the form*

$$\int \frac{p(x)q'(x)}{(q(x))^2} dx$$

for some differentiable functions p and q, then we may write

$$\int \frac{p(x)q'(x)}{(q(x))^2} dx = \frac{-p(x)}{q(x)} + \int \frac{p'(x)}{q(x)} dx.$$

We can use Theorem 3 to solve an integral that's already in Stewart's textbook!

Example 1 ([5] Chapter 6.1, Exercise #15).

$$\int \frac{xe^{2x}}{(1+2x)^2} dx.$$

Since the denominator is written as something squared, we have a clear choice for q. From there, we calculate q' to see what p will have to be.

$$q = 1 + 2x \qquad p = \frac{1}{2} xe^{2x}$$

$$q' = 2dx \quad p' = \frac{1}{2}(e^{2x} + 2xe^{2x})dx.$$

Then we apply Theorem (3) to get:

$$\begin{aligned}
\int \frac{xe^{2x}}{(1+2x)^2}dx =& \frac{-\frac{1}{2}xe^{2x}}{1+2x} + \int \frac{\frac{1}{2}(e^{2x}+2xe^{2x})}{1+2x}dx \\
=& -\frac{1}{2}\cdot\frac{xe^{2x}}{1+2x} + \frac{1}{2}\int \frac{e^{2x}+2xe^{2x}}{1+2x}dx \\
=& \frac{-xe^{2x}}{2(1+2x)} + \frac{1}{2}\int \frac{e^{2x}(1+2x)}{1+2x}dx \\
=& \frac{-xe^{2x}}{2(1+2x)} + \frac{1}{2}\int e^{2x}dx \\
=& \boldsymbol{\frac{-xe^{2x}}{2(1+2x)} + \frac{e^{2x}}{4} + C}.
\end{aligned}$$

If we find a common denominator, then we can simplify this further, and get $\frac{1}{4}\cdot\frac{e^{2x}}{1+2x}+C$.

And thus, as was first pointed out to me by the YouTuber blackpenredpen, the "original function" was a quotient the whole time![3][2]

2. Integration by Weirder Parts

Since Example 1 is an exercise in the Integration by Parts section of Stewart's book ([5]), he expects us to solve it using Integration by Parts. There are multiple ways to use Integration by Parts that will work, and I have written these out in detail in Appendix B. I think that my first solution is Stewart's intended solution. Choose

$$\begin{aligned}
f &= xe^{2x} & g' &= \frac{1}{(1+2x)^2}dx \\
f' &= (2xe^{2x}+e^{2x})dx & g &= \frac{-1}{2(1+2x)},
\end{aligned}$$

$$\text{so } \int \frac{xe^{2x}}{(1+2x)^2}dx = \frac{-xe^{2x}}{2(1+2x)} - \int \frac{-(e^{2x}+2xe^{2x})}{2(1+2x)}dx.$$

At this point, we would solve the same integral as before, so I will stop here.

Notice that our choice of p is almost identical to our choice for f. They only differ by a constant. This is no coincidence. Just as we can prove the Quotient Rule using the Product Rule and Chain Rule,[3], we can prove the Inverse Quotient Rule using Integration by Parts and a u-Substitution:

[2]blackpenredpen (a.k.a. Steve Chow) actually solved this integral in a previous video, [2], by "inspection". In other words, he guessed the correct answer, and then proved it was correct by differentiating it!

[3]Write $\frac{p}{q} = pq^{-1}$ then differentiate both sides.

Proof (Theorem 3). Start with an integral of the form $\int \frac{p(x)q'(x)}{(q(x))^2}dx$. We will use Integration by Parts, with the following choices for f and g':

$$f = p(x) \qquad g' = \frac{q'(x)}{q(x)^2}dx$$

$$f' = p'(x)dx \qquad g = \frac{-1}{q(x)},$$

using the substitution $u = q(x)$ on the g' piece. Then we write

$$\begin{aligned}\int \frac{p(x)q'(x)}{(q(x))^2}dx =& p(x) \cdot \frac{-1}{q(x)} - \int p'(x) \cdot \frac{-1}{q(x)}dx \\ =& \frac{-p(x)}{q(x)} + \int \frac{p'(x)}{q(x)}dx.\end{aligned}$$

□

If this is the case, then why would we ever use this strategy? Well, we could just as easily ask why we bother learning the Quotient Rule when we learn how to differentiate. Can't we just turn all of our quotients into products and forget about fractions? In my experience, this doesn't happen. Students choose to memorize the Quotient Rule and don't bother taking an extra step, because the extra algebraic manipulation isn't worth it to them.

2.1. Algebraic Manipulation

Some integrals can be solved with algebraic manipulation if we insist on sticking with the rules we already know. For example, when I teach Calculus 1 at the University of Pittsburgh, students learn to solve integrals such as the following:

Example 2 ([1] LON-CAPA integration-2, #3.1)**.**

$$\int \frac{3}{x^2+5}dx.$$

Calculus 1 students at the University of Pittsburgh do not always learn Trig Substitution. To handle an integral like this, they are taught that since the derivative of $\arctan(x)$ is $\frac{1}{1+x^2}$, they should try to make the above integral look something like $\int \frac{1}{1+\diamondsuit^2}dx$, and then use the substitution $u = \diamondsuit$. In this example, we could factor a 3 out of the numerator and a 5 out of the denominator to get

$$\begin{aligned}\int \frac{3}{x^2+5}dx =& \frac{3}{5}\int \frac{1}{1+\frac{x^2}{5}}dx \\ =& \frac{3}{5}\int \frac{1}{1+\left(\frac{x}{\sqrt{5}}\right)^2}dx\end{aligned}$$

$$
\begin{aligned}
&= \frac{3}{5} \int \sqrt{5} \cdot \frac{1}{1 + \diamondsuit^2} d\diamondsuit \\
&= \frac{3\sqrt{5}}{5} \arctan\left(\frac{x}{\sqrt{5}}\right) + C.
\end{aligned}
$$

However, as soon as you learn Trig Substitution, you have a more straightforward path to the solution with the substitution $x = \sqrt{5}\tan(\theta)$. Most of that algebraic manipulation is unnecessary!

We can look at a similar example for the Inverse Quotient Rule. If we use u-Substitution, Partial Fraction Decomposition, or Integration by Parts on the following integral, we will need to do some algebra. Alternatively, we could just use the Inverse Quotient Rule:

Example 3 (An Integral with Many Strategies)**.**

$$
\int \frac{10x}{(2x-3)^2} dx.
$$

Choose

$$
\begin{aligned}
q &= 2x - 3 \quad & p &= 5x \\
q' &= 2 \quad & p' &= 5.
\end{aligned}
$$

So we have

$$
\begin{aligned}
\int \frac{10x}{(2x-3)^2} dx &= \frac{-5x}{2x-3} + \int \frac{5}{2x-3} dx \\
&= \boldsymbol{\frac{-5x}{2x-3} + \frac{5}{2} \ln|2x-3| + C}.
\end{aligned}
$$

There is some value in comparing this to the solution you would get if you used a u-Substitution, so let's try it that way.

Let $u = 2x - 3$. Then $du = 2dx$, and we can write $x = \frac{1}{2}(u+3)$. So we get

$$
\begin{aligned}
\int \frac{10x}{(2x-3)^2} dx &= \frac{1}{2} \int \frac{10\left(\frac{1}{2}(u+3)\right)}{u^2} du \\
&= \frac{5}{2} \int \frac{u+3}{u^2} du \\
&= \frac{5}{2} \int \frac{u}{u^2} du + \int \frac{3}{u^2} du \\
&= \frac{5}{2}\left(\ln|u| - \frac{3}{u}\right) + C \\
&= \boldsymbol{\frac{5}{2} \ln|2x-3| - \frac{15}{2(2x-3)} + C}.
\end{aligned}
$$

These two answers look fairly distinct. Note that when you calculate indefinite integrals in different ways, you may get answers which differ by a constant. To show that the second answer is equivalent to the first, subtract $\frac{5}{2}$ from the second answer:

$$\begin{aligned}
&\frac{5}{2}\ln|2x-3| - \frac{15}{2(2x-3)} - \frac{5}{2}\\
=&\frac{5}{2}\ln|2x-3| - \frac{15}{2(2x-3)} - \frac{5(2x-3)}{2(2x-3)}\\
=&\frac{5}{2}\ln|2x-3| - \frac{15-10x+15}{2(2x-3)}\\
=&\boldsymbol{\frac{5}{2}\ln|2x-3| + \frac{-5x}{2x-3}}.
\end{aligned}$$

3. The Structure of the Parts

3.1. We Can Make Integration by Parts Easier

When students first learn Integration by Parts, they might be taught an acronym, called LIATE, to help them choose the parts appropriately. If they do not know how to decompose the integral into f and g' pieces, they are advised to work their way down this list:

Choose for f	Function
L	Logarithms
I	Inverse (Trig)
A	Algebraic[4]
T	Trig
E	Exponential

For the majority of Integration by Parts integrals that calculus students see, choosing which function to differentiate based on this hierarchy will be a helpful first step. Typically, this will help you turn a difficult integral into an easier integral. However, Example 1 is one of many exceptions to this rule!

Starting with $\int \frac{xe^{2x}}{(1+2x)^2}dx$, if we were to use the LIATE strategy, then we would scan down our list to choose f. There are no Logarithmic functions and no Inverse Trig functions, so we would choose f to be an Algebraic function, and g' will be everything else:

$$\begin{aligned}
f &= \frac{x}{(1+2x)^2}dx & g' &= e^{2x}dx\\
f' &= \frac{(1+2x)^2 - 4x(1+2x)}{(1+2x)^4}dx & g &= \frac{1}{2}e^{2x},
\end{aligned}$$

[4]Algebraic functions are functions that only use addition, subtraction, multiplication, division, and raising things to a fractional power. In a high school Algebra class, students learn how to work with polynomials, which are a special kind of algebraic function!

When we apply our Integration by Parts formula, we're clearly going to end up with a more complicated integral. As we saw earlier, the best choice for f is xe^{2x}, which does not follow the LIATE strategy. On the other hand, when we used the Inverse Quotient Rule, we immediately decomposed the integral properly, and those parts also corresponded to the best choice for Integration by Parts!

3.2. Higher Powers in the Denominator

Consider the following integral:

Example 4 (A Tricky Trig Integral)**.**

$$\int \frac{x\sec^2(x)}{\tan^4(x)}dx.$$

If we wanted to use the Inverse Quotient Rule on this integral, it looks like we should choose $q = \tan^2(x)$. Let's see what happens:

$$\begin{aligned} q &= \tan^2(x) & p &= \frac{1}{2}\cdot\frac{x}{\tan(x)} \\ q' &= 2\tan(x)\sec^2(x) & p' &= \frac{1}{2}\cdot\frac{\tan(x) - x\sec^2(x)}{\tan^2(x)}. \end{aligned}$$

Then we apply Theorem (3) to get:

$$\int \frac{x\sec^2(x)}{\tan^4(x)}dx = -\frac{1}{2}\cdot\frac{x}{\tan^3(x)} + \int \frac{1}{2}\cdot\frac{\tan(x) - x\sec^2(x)}{\tan^4(x)}dx.$$

It looks like we didn't make our integral much easier![5] Notice that when we let $q = \tan^2(x)$, there are also copies of $\tan(x)$ in q'. Then we need to account for these when we solve for p. Hence, we ended up with a quotient, which made our derivative fairly complicated.

Ideally, we could choose $q(x)$ in such a way that the entire denominator is accounted for without creating issues in the numerator. And if the problem is that we're seeing more copies of $\tan(x)$ in the numerator, maybe we could put some extra $\tan(x)$'s there to address that:

$$\int \frac{x\sec^2(x)}{\tan^4(x)}dx = \int \frac{x\sec^2(x)}{\tan^4(x)}\cdot\frac{\tan^2(x)}{\tan^2(x)}dx = \int \frac{x\sec^2(x)\tan^2(x)}{\tan^6(x)}dx$$

Since we've multiplied by $\tan^2(x)$ in the numerator and denominator, we've changed our denominator, and so we'll have $q = \tan^3(x)$ instead:

[5]There is still a way forward from here, but it'll require some tricks and more work, so I do not recommend this strategy.

$$q = \tan^3(x) \qquad p = \frac{x}{3}$$

$$q' = 3\tan^2(x)\sec^2(x) \quad p' = \frac{1}{3}.$$

Then we apply Theorem (3) to get:

$$\begin{aligned}
\int \frac{x\sec^2(x)}{\tan^4(x)}dx =& \frac{-x}{3\tan^3(x)} + \int \frac{1}{3\tan^3(x)}dx \\
=& \frac{-x}{3\tan^3(x)} + \frac{1}{3}\int \cot^3(x)dx \\
=& \frac{-x}{3\tan^3(x)} + \frac{1}{3}\int \cot(x)(\csc^2(x)-1)dx \\
=& \frac{-x}{3\tan^3(x)} + \frac{1}{3}\int \cot(x)\csc^2(x)dx - \frac{1}{3}\int \cot(x)dx \\
=& \frac{-x}{3\tan^3(x)} + \frac{1}{3}\int \cot(x)\csc^2(x)dx - \frac{1}{3}\int \frac{\cos(x)}{\sin(x)}dx \\
=& \boldsymbol{\frac{-x}{3\tan^3(x)} - \frac{\cot^2(x)}{6} - \frac{\ln|\sin(x)|}{3} + C}.^{6}
\end{aligned}$$

It took an extra algebraic step, but we managed to make this integral easier. Fortunately, there is a simple rule for choosing q when the power in the denominator is larger than 2, which we can apply without using any algebra:

Theorem 4 (The Generalized Inverse Quotient Rule)**.** *If we can write an integral in the form*

$$\int \frac{P(x)Q'(x)}{(Q(x))^m}dx$$

for some differentiable functions P and Q and some number $m \neq 1$, then we may write

$$\int \frac{P(x)Q'(x)}{(Q(x))^m}dx = \frac{1}{m-1}\left[\frac{-P(x)}{Q(x)^{m-1}} + \int \frac{P'(x)}{Q(x)^{m-1}}dx\right].$$

Proof. To more clearly see how the algebra will work out, multiply the numerator and denominator by $(Q(x))^{m-2}$:

$$\int \frac{P(x)Q'(x)}{(Q(x))^m}dx = \int \frac{P(x)Q'(x)Q(x)^{m-2}}{(Q(x))^{2m-2}}dx.$$

Now apply the Inverse Quotient Rule:

[6] For the first integral, use the substitution $u = \cot(x)$, and for the second integral, use the substitution $u = \sin(x)$. If you'd like some more context on the strategy to solve $\int \cot^3(x)dx$, see Chapter 6.2 of Stewart's textbook [5].

$$q = (Q(x))^{m-1} \qquad p = \frac{P(x)}{m-1}$$

$$q' = (m-1)(Q(x))^{m-2}Q'(x) \quad p' = \frac{P'(x)}{m-1}.$$

Thus, we get

$$\begin{aligned} \int \frac{P(x)Q'(x)Q(x)^{m-2}}{(Q(x))^{2m-2}}dx =& \frac{-P(x)}{(m-1)(Q(x))^{m-1}}dx + \int \frac{P'(x)}{(m-1)(Q(x))^{m-1}}dx \\ =& \frac{1}{m-1}\left[\frac{-P(x)}{(Q(x))^{m-1}}dx + \int \frac{P'(x)}{(Q(x))^{m-1}}dx\right]. \end{aligned}$$

□

Put simply, if our denominator looks like $(Q(x))^m$, we should choose $q = (Q(x))^{m-1}$. Theorem 4 is typically useful when m is an integer, but it will work as long as $m \neq 1$.

3.3. A Second Inverse Quotient Rule?

So far, we have been using the formula

$$\int \frac{pq'}{q^2} = -\frac{p}{q} + \int \frac{p'}{q}. \tag{1}$$

Notice that, unlike Integration by Parts, this is not a symmetric formula. We could have just as easily solved for the other integral:

$$\int \frac{p'}{q} = \frac{p}{q} + \int \frac{pq'}{q^2}. \tag{2}$$

In other words, there are two ways to use the Inverse Quotient Rule, and each allows us to swap one integral for another. So far, we have been using Formula 1, but Formula 2 is an equally valid formula!

Theorem 5 (The Inverse Quotient Rule, Version 2)**.** *If we can write an integral in the form*

$$\int \frac{p'(x)}{q(x)}$$

for some differentiable functions p and q, then we may write

$$\int \frac{p'(x)}{q(x)}dx = \frac{p(x)}{q(x)} + \int \frac{p(x)q'(x)}{(q(x))^2}dx.$$

When we used Theorem 3 in earlier examples, we turned a difficult integral into an easier integral. If Theorem 5 is going to be useful to us, then it should be able to do likewise. This is a very general problem-solving strategy, and it is an excellent way to think about solving integrals: "How can I make what I have in front of me a little bit easier to solve?"

You will not be surprised, then, that examples where Theorem 5 actually help us solve an integral are tougher to come up with. Under what circumstances would a function in the form $\frac{pq'}{q^2}$ be easier to integrate than $\frac{p'}{q}$?

One way this might happen is if certain terms in the numerator and denominator of $\frac{pq'}{q^2}$ could cancel with each other. Let's investigate some potential ways this could happen:

- If $p = q$, then we could write the original integral as $\int \frac{p'}{q} = \int \frac{p'}{p}$, which can be solved with the u-Substitution $u = p$.

- If $p = q^2$, then $p' = 2qq'$ by the Chain Rule. Thus, we would rewrite

  $$\int \frac{p'}{q} = \int \frac{2qq'}{q} = \int 2q',$$

  so our original structure would never occur organically in an integral.

- If $q^2 = kq'$ for some number k, then we can find q by solving this differential equation.[7] You will get $q = \frac{1}{C - kx}$ for some constant C. When we rewrite the integral, this will give us

  $$\int \frac{p}{\frac{1}{C-kx}} = \int p \cdot (C - kx).$$

  You would not think to use the Inverse Quotient Rule on this because we no longer have a quotient. Instead, you'd likely try Integration by Parts.

- If $q^2 = kpq'$ for some number k, then we can find q by solving another differential equation.[8] You will get

  $$q = \frac{-k}{C + \int \frac{1}{p}}$$

  for some constant C. Note that p can no longer be any function, since we can't always integrate $\frac{1}{p}$. However, like the previous case, we would never see an integral where q actually looks like this, and you'd most likely try Integration by Parts.

- If $q = kq'$ for some number k, then we can find q by solving yet another differential equation.[9] You will get $q = Ae^{x/k}$ for some constant A. When we rewrite our

[7] If you don't know how to solve differential equations, you can treat this result as a black box without missing out on anything else in this paper.

[8] See footnote 7.

[9] See footnote 7.

integral, we get

$$\int \frac{p'}{q} = \int \frac{p'}{Ae^{x/k}},$$

and by applying Theorem 5, we get

$$\int \frac{p'}{Ae^{x/k}} = \frac{p}{Ae^{x/k}} + \int \frac{\frac{A}{k}pe^{x/k}}{Ae^{2x/k}} = \frac{p}{Ae^{x/k}} + \frac{1}{k}\int \frac{p}{e^{x/k}}.$$

This integral will sometimes be solvable, but it will rarely be easier to solve than our original integral. Given the choice, would you rather have the function p in the numerator of this integral, or its derivative p'?

- Finally, if $q = kpq'$ for some number k, then we can find q by solving one last differential equation.[10] You will get

$$q = Ae^{\frac{1}{k}\int \frac{1}{p}dx}$$

for some constant C. Once again, p cannot be just any function. When we rewrite our integral, we get

$$\int \frac{p'}{q} = \int \frac{p'}{Ae^{\frac{1}{k}\int \frac{1}{p}dx}},$$

and by applying Theorem 5, we get

$$\begin{aligned}\int \frac{p'}{Ae^{\frac{1}{k}\int \frac{1}{p}dx}} &= \frac{p}{Ae^{\frac{1}{k}\int \frac{1}{p}dx}} + \int \frac{\frac{A}{kp}pe^{\frac{1}{k}\int \frac{1}{p}dx}}{e^{\frac{2}{k}\int \frac{1}{p}dx}} \\ &= \frac{p}{Ae^{\frac{1}{k}\int \frac{1}{p}dx}} + \frac{A}{k}\int \frac{1}{e^{\frac{1}{k}\int \frac{1}{p}dx}} \\ &= \frac{p}{Ae^{\frac{1}{k}\int \frac{1}{p}dx}} + \frac{A}{k}\int e^{\frac{-1}{k}\int \frac{1}{p}dx}.\end{aligned}$$

This integral will almost never be solvable, and in most of the solvable cases, p' will be a fraction. One exception is when $p = x$, but we can investigate this integral quickly. One possible value for q (where $k = A = 1$) is $q = e^{\ln(x)} = x$, which would give us

$$\int \frac{p'}{q}dx = \int \frac{1}{x}dx = \ln(x) + C.$$

That makes our Inverse Quotient Rule look a bit silly!

In any case, because q will typically be an exponential function, we can often move it to the numerator, and we would likely not think to use the Inverse Quotient Rule here.

This is not a thorough exploration, but it does not seem like we should expect specific pieces of our integral to cancel out in a way that will help us. And without any

[10]See footnote 7.

cancellation, it doesn't seem likely that Theorem 5 will be useful to us.

4. Conclusion

When I began writing this paper, I was convinced that the Inverse Quotient Rule would be a very niche strategy. However, as I started to come up with more examples, the Inverse Quotient Rule started looking increasingly powerful and useful, and it occasionally has important advantages over other strategies.

First, the Generalized Inverse Quotient Rule can make quick work of certain integrals which would otherwise be quite tedious by Substitution or Partial Fraction Decomposition. Second, for integrals where we'd usually use Integration by Parts, the Inverse Quotient Rule can give us a good set of parts right away, even in cases where the LIATE strategy doesn't work. Finally, the Inverse Quotient Rule produces a strategy to solve fractional Trig Integrals, which are not studied in Stewart's textbook [5]. I have included many of these among the exercises in Appendix A.

The Inverse Quotient Rule is not an all-powerful strategy, but it has its moments in the spotlight. Many of the exercises I've included will ask you to compute an integral in multiple ways, which will help show strengths and weaknesses of the Inverse Quotient Rule when compared to other methods. A few of the exercises require some knowledge of Trig Integrals, Trig Substitutions, and/or Partial Fraction Decomposition, and any bold numbers indicate especially interesting or challenging integrals.

Just as the Quotient Rule is seen as an important concept in our theory of derivatives, the Inverse Quotient Rule should be incorporated into our theory of integrals!

References

[1] Angela Athanas. *University of Pittsburgh: MATH 0220 ANALYTC GEOMETRY & CALCULUS 1*. integration-2, #3.1. 2006.

[2] Steve Chow. *fun integral battle#7: you might as well do them this way!* `https://www.youtube.com/watch?v=rXgcSp3Lisw`. 2016.

[3] Steve Chow. *integral of* $xe^{2x}/(1+2x)^2$*, LIATE DOESN'T WORK HERE.* `https://www.youtube.com/watch?v=SSPb1CtLLxI`. 2017.

[4] Steve Chow. *Why I Don't Teach LIATE (integration by parts trick).* `https://www.youtube.com/watch?v=z33rYYUM4d8&t=547s`. 2021.

[5] James Stewart. *Essential Calculus, Early Transcendentals, Second Edition*. Cengage Learning, 2012.

A. Inverse Quotient Rule Exercises

Calculate the following integrals once by starting with a u-Substitution, and once by starting with the Inverse Quotient Rule. Which approach was easier?

1. $\displaystyle\int_{\pi/3}^{\pi/2} \frac{\cos^3(x)}{\sin^2(x)} dx$
2. $\displaystyle\int \frac{\sec^4(x)}{\tan^2(x)} dx$
3. $\displaystyle\int_2^4 \frac{x}{(2x^2-5)^2} dx$
4. $\displaystyle\int \frac{\cos(x)}{\sin^4(x)} dx$
5. $\displaystyle\int \frac{x^7+2x^3}{(x^4+8)^2} dx$

6. $\displaystyle\int_{\pi/6}^{\pi/3} \frac{\sin^6(x)}{\cos^3(x)} dx$

7. $\displaystyle\int \frac{\csc^4(x)}{\cot^3(x)} dx$

Set up strategies to solve the following integrals in three different ways:

- a u-Substitution,
- a Partial Fraction Decomposition,
- the Inverse Quotient Rule.

Then calculate the integral using whichever strategy looks most promising.

8. $\displaystyle\int \frac{x^4}{(2x^3-5)^2} dx$
9. $\displaystyle\int_1^2 \frac{x^3}{(x-3)^4} dx$

10. $\displaystyle\int \frac{x^m}{(x+1)^n} dx$, for pos. ints. $m < n$.
(Use sum notation in your answer!)

11. $\displaystyle\int_0^1 \frac{x^5}{(x-2)^9(x+2)^9} dx$
12. $\displaystyle\int \frac{x^2}{(x+1)^7(x-1)^7} dx$

Begin with Integration by Parts to solve the following integrals. If you get stuck, begin with the Inverse Quotient Rule instead, and then use that to find the correct parts for an Integration by Parts step. In this first step, does the best choice for Integration by Parts follow the LIATE strategy?

13. $\displaystyle\int \frac{x^3e^{-4x}}{(3-4x)^2} dx$
14. $\displaystyle\int \frac{x\cos(x)}{\sin^2(x)} dx$
15. $\displaystyle\int \frac{\ln(2x+7)}{(2x+7)^2} dx$
16. $\displaystyle\int_0^3 \frac{xe^{3x/2}}{(3x+2)^2} dx$
17. $\displaystyle\int \frac{x^2e^x}{(x+2)^2} dx$
18. $\displaystyle\int_{\pi/4}^{\pi/3} \frac{\sec^3(x)}{\tan^2(x)} dx$
19. $\displaystyle\int \frac{x^2\arctan(x)}{(x^2+1)^2} dx$

20. $\displaystyle\int_1^e \frac{\ln(x)}{(1+\ln(x))^2} dx$
(Credit to [4] for this integral!)

Calculate the following integrals:

21. $\displaystyle\int \frac{x^6}{(x^2-3)^4} dx$

22. $\displaystyle\int \frac{x\ln(x)}{(x^2-1)^2} dx$

23. $\displaystyle\int \frac{\cos^6(x)}{\sin^4(x)} dx$

24. $\displaystyle\int \frac{x^ae^{bx}}{(bx+a)^2} dx$

25. $\displaystyle\int \frac{x^3\ln(x^2+4)}{(x^2+4)^2} dx$

26. $\displaystyle\int_0^{1/2} \frac{x^2\arcsin(x)}{(1-x^2)^2} dx$

B. Integration by Parts Solutions to Example 1

Calculate the integral $\int \frac{xe^{2x}}{(1+2x)^2}dx$.

SOLUTION 1

We will compute this integral using Integration by Parts, and the following choices for f and dg:

$$\begin{aligned} f &= xe^{2x} & g &= \frac{1}{(1+2x)^2}dx \\ df &= (2xe^{2x}+e^{2x})dx & g &= \frac{1}{2}\left(\frac{-1}{1+2x}\right). \end{aligned}$$

Given dg, we can find g using the substitution $u = 1+2x$. Also note that we can factor e^{2x} out of our df, giving us $df = (2xe^{2x}+e^{2x})dx = e^{2x}(1+2x)dx$. We can then calculate the integral:

$$\begin{aligned} &\int \frac{xe^{2x}}{(1+2x)^2}dx \\ =&\frac{-1}{2}\cdot\frac{xe^{2x}}{1+2x} - \int \frac{-1}{2}\cdot\frac{e^{2x}(1+2x)}{(1+2x)}dx \\ =&\frac{-1}{2}\cdot\frac{xe^{2x}}{1+2x} + \int \frac{1}{2}e^{2x}dx \\ =&\boldsymbol{\frac{-xe^{2x}}{2(1+2x)} + \frac{e^{2x}}{4} + C}. \end{aligned}$$

You could plug in the bounds of integration now and be done, but this integral simplifies very nicely:

$$\begin{aligned} &\frac{-xe^{2x}}{2(1+2x)} + \frac{e^{2x}}{4} + C \\ &= \frac{-2xe^{2x}}{4(1+2x)} + \frac{e^{2x}(1+2x)}{4(1+2x)} + C \\ &= \frac{-2xe^{2x}}{4(1+2x)} + \frac{e^{2x}+2xe^{2x}}{4(1+2x)} + C \\ &= \frac{-2xe^{2x}+e^{2x}+2xe^{2x}}{4(1+2x)} + C \\ &= \boldsymbol{\frac{1}{4}\cdot\frac{e^{2x}}{1+2x} + C}. \end{aligned}$$

(See SOLUTION 2 on the next page)

Solution 2

You can choose differently for f and dg, but it's *very* tricky! Make the following choices for f and dg:

$$f = e^{2x} \qquad dg = \frac{x}{(1+2x)^2}dx$$
$$df = 2e^{2x}dx \quad g = \frac{1}{4}\left(\ln|1+2x| + \frac{1}{1+2x}\right).$$

Given dg, we can find g in one of two ways. One way is by using Partial Fraction Decomposition; the resulting integral will be $\int\left(\frac{1/2}{1+2x} - \frac{1/2}{(1+2x)^2}\right)dx$, which is solved using the substitution $u = 1+2x$. The other way is to use the substitution $u = 1+2x$ immediately, making sure to also replace $x = \frac{u-1}{2}$ in the numerator. Then you can split up the resulting fraction: $\frac{1}{2}\int\frac{\frac{u-1}{2}}{u^2}du = \frac{1}{4}\left(\int\frac{u}{u^2}du - \int\frac{1}{u^2}du\right)$.

Whichever method you use, this gives us the following integral:

$$\begin{aligned}
&\int\frac{xe^{2x}}{(1+2x)^2}dx\\
=&\frac{1}{4}e^{2x}\left(\ln|1+2x| + \frac{1}{1+2x}\right) - \int\frac{1}{4}\cdot 2e^{2x}\left(\ln|1+2x| + \frac{1}{1+2x}\right)dx\\
=&\frac{1}{4}e^{2x}\left(\ln|1+2x| + \frac{1}{1+2x}\right)\boxed{-\frac{1}{2}\int e^{2x}\ln|1+2x|dx} - \frac{1}{2}\int\frac{e^{2x}}{1+2x}dx.
\end{aligned}$$

This looks concerning, since we can't solve the integral in the box. But let's work through the integral on the right before we give up. We'll need to use Integration by Parts again:

$$f = e^{2x} \qquad dg = \frac{1}{1+2x}dx$$
$$df = 2e^{2x}dx \quad g = \frac{1}{2}\ln|1+2x|.$$

(To solve for g, use the u-substitution $u = 1+2x$.)

And so we have

$$\begin{aligned}
&\frac{1}{4}e^{2x}\left(\ln|1+2x| + \frac{1}{1+2x}\right) - \frac{1}{2}\int e^{2x}\ln|1+2x|dx - \frac{1}{2}\int\frac{e^{2x}}{1+2x}dx\\
=&\frac{1}{4}e^{2x}\left(\ln|1+2x| + \frac{1}{1+2x}\right) - \frac{1}{2}\int e^{2x}\ln|1+2x|dx - \frac{1}{2}\left(\frac{1}{2}e^{2x}\ln|1+2x| - \int e^{2x}\ln|1+2x|dx\right)\\
=&\frac{1}{4}e^{2x}\left(\ln|1+2x| + \frac{1}{1+2x}\right)\boxed{-\frac{1}{2}\int e^{2x}\ln|1+2x|dx} - \frac{1}{4}e^{2x}\ln|1+2x| + \boxed{\frac{1}{2}\int e^{2x}\ln|1+2x|dx}\\
=&\frac{1}{4}e^{2x}\left(\ln|1+2x| + \frac{1}{1+2x}\right) - \frac{1}{4}e^{2x}\ln|1+2x| + C\\
=&\frac{1}{4}e^{2x}\ln|1+2x| + \frac{1}{4}\cdot\frac{e^{2x}}{1+2x} - \frac{1}{4}e^{2x}\ln|1+2x| + C\\
=&\boldsymbol{\frac{1}{4}\cdot\frac{e^{2x}}{1+2x} + C}.
\end{aligned}$$

(See Solution 3 on the next page)

Solution 3 (Based on a solution by my student, Chubo Wang, Summer 2022)

Let's first make the u-substitution $u = 1 + 2x$. Note that $du = 2dx$, $2x = u - 1$, and $x = \frac{u-1}{2}$. And so

$$\begin{aligned}
&\int \frac{xe^{2x}}{(1+2x)^2}dx \\
=&\frac{1}{2}\int \frac{\left(\frac{u-1}{2}\right)e^{u-1}}{u^2}du \\
=&\frac{1}{4}\int \frac{(u-1)\frac{e^u}{e}}{u^2}du \\
=&\frac{1}{4e}\int \frac{(u-1)e^u}{u^2}du \\
=&\frac{1}{4e}\left(\int \frac{ue^u}{u^2}du - \int \frac{e^u}{u^2}du\right) \\
=&\frac{1}{4e}\left(\boxed{\int \frac{e^u}{u}du} - \int \frac{e^u}{u^2}du\right)
\end{aligned}$$

Similarly to Solution 2, we find that we cannot solve the integral in the box. However, we can use Integration by Parts on the second integral to cancel out the first integral. Make the following choices for f and dg:

$$f = e^u \qquad dg = \frac{1}{u^2}du$$

$$df = e^u du \qquad g = \frac{-1}{u}.$$

Thus, we have:

$$\begin{aligned}
&\frac{1}{4e}\left(\int \frac{e^u}{u}du - \int \frac{e^u}{u^2}du\right) \\
=&\frac{1}{4e}\left(\int \frac{e^u}{u}du - \left(\frac{-e^u}{u} - \int \frac{-e^u}{u}du\right)\right) \\
=&\frac{1}{4e}\left(\boxed{\int \frac{e^u}{u}du} + \frac{e^u}{u}\boxed{-\int \frac{e^u}{u}du}\right) \\
=&\frac{1}{4e}\cdot\frac{e^u}{u} + C \\
=&\frac{1}{4e}\cdot\frac{e^{1+2x}}{1+2x} + C \\
=&\frac{1}{4e}\cdot\frac{e\cdot e^{2x}}{1+2x} + C \\
=&\boldsymbol{\frac{1}{4}\cdot\frac{e^{2x}}{1+2x} + C}.
\end{aligned}$$