Pittsburgh Interdisciplinary Mathematics Review
Volume 1, Number 1, April 2024, Pages 24–32
https://doi.org/10.5195/pimr.2024.19

What Are the Different Infinities?

A Mathematical Odyssey

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1. Introduction

To infinity and beyond! These famous words from our beloved space ranger capture the essence of the journey we are to take into the limitless realm of mathematics.

Whether or not you have studied upper-level mathematics, you have most likely heard of infinity. Endless, boundless, and enormous are some words that may come to your mind when you think of the term. We use the term as a concept rather than a specific number.

However, did you know that there are different kinds of infinities, some bigger than others?

German mathematician Georg Cantor was one of the first explorers of infinity. He utilized sets to venture into the infinite, uncovering some truly fascinating results. On our voyage, we will visit two of Cantor's most intriguing findings. As a consequence of Cantor's Theorem, we will initially come across the intriguing concept that there is no "largest" set. Additionally, we will unveil the existence of different kinds of infinity, where one infinity is greater than the other, and the uncountable nature of real numbers.

2. Historical Background

Infinity has always been considered, in a numerical sense, a very large number. There had been a dispute among mathematicians on whether infinity can be used in rigorous mathematics. Many believed that the concept was more philosophical than mathematical. Mathematicians of the time also made a distinction between potential infinity and actual infinity. The set of natural numbers $\{0, 1, 2, 3, ...\}$ was considered potentially infinite. This was because while each number has a successor and you could never count all the naturals, at any given point while counting, only a finite number of naturals would have been counted. This makes it potentially infinite but not infinite in actuality.

However, let's consider the integers in ascending order: $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$. Notice that at any point, there are an infinite number of integers before it. Thus, past mathematicians referred to the set of integers as actually infinite. The few mathematicians that did indeed support the notion of infinity were referring to the sets that are potentially infinite and not those that are actually infinite.

Quoting Aristotle's physics, "The infinite has a potential existence... There will not be an actual infinite." A more modern mathematician, Carl Friedrich Gauss, expressed a similar view in a letter to Schumacher dated 1831: "I must vehemently protest against your use of the infinite as something consummated, as this is never permitted in mathematics. The infinite is but a façon de parler, meaning a limit to which certain ratios may approach as closely as desired when others are permitted to increase indefinitely." The French phrase "façon de parler" translates to "figure of speech." In this context, Gauss says that infinity is more of a figure of speech used to describe large quantities in general and not a well-defined number. This was Gauss' response to those mathematicians who would use infinity just as you would an ordinary number. Widely regarded as one of the brightest mathematicians, even Euler had treated infinity in such a way. He had stated that $1/0$ is infinite and $2/0$ is twice as large as $1/0$.

However, Cantor challenged the views held by most mathematicians up until the 19th century. He accepted both potential and actual infinity. Notice that the set of integers could be rewritten as $\{0, -1, 1, -2, 2, ...\}$ (the order of elements is irrelevant in a set). This way, it would be the same type of infinity as the set of natural numbers. Hence, making the distinction between actual and potential infinity moot. Cantor also went ahead and showed that there is a different way to classify the infinities: enumerable (countable) and non-enumerable (uncountable).

Expressing such views was nothing short of an act of rebellion by Cantor, as his ideas were in direct opposition to those held by the brightest minds of his time. Thus, his ideas were widely criticized and even mocked at the time.[2]

3. A Primer on Sets and Functions

First, we must familiarize ourselves with some of the tools we will use in our exploration. This will ensure that our voyage is as pleasant and comfortable of an experience as possible.

A **set** is a collection of things. These things could be planets, animals, countries, etc., but in mathematics, they are usually numbers. A set can also be a collection of other sets. The things in a set are called its members or **elements**. We write $x \in A$ to denote that x is an element of the set A.

We can represent a set by listing out its elements. For example: $B = \{2, 3, 5\}$. We can also represent a set by a property. All things that satisfy the property would be elements of the set. For example: The set $C = \{x \mid x \text{ is a prime number less than } 6\}$ would contain the elements 2, 3 and 5.

A set Y is a **subset** of set Z if and only if all the elements of set Y are members of set Z. If $Y = Z$, then Y is an improper subset of Z and all other subsets of Z are proper. $Y \subset Z$ denotes that Y is a subset of Z. Note that the empty set $\{\}$, the only set with no elements, is a subset of all sets.

Let $A = \{1, 2, 3, 4\}$ and $B = \{3, 4\}$. $B \subset A$ since all the elements of B, namely 3 and 4, are also members of A. In fact, B is a proper subset of A.

The set of all such subsets, proper and improper, of A is called the **Power Set** of A, denoted as $P(A)$. In our example,

$$P(A) = \{\{\}, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}, \{1,2,3\}, \{2,3,4\}, \{1,3,4\}, \{1,2,4\}, \{1,2,3,4\}\}$$

The **cardinality** of a set is the number of elements in it. The cardinality of the set A is 4, and the cardinality of the empty set is 0. $|A|$ denotes the cardinality of set A.

Now that we have covered the basics of sets, let us move on to functions. A **function** defines a mapping between two sets where every element in a set is uniquely mapped to an element in the other set.

Consider the followings sets: $K = \{a, b, c\}$ and $M = \{x, y, z\}$. Let us define a function f between K and M as in Figure 1.

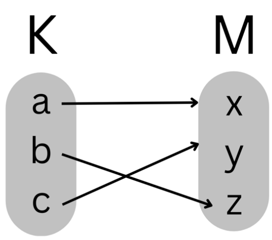

Figure 1: Bijective Function $f : K \xrightarrow{\sim} M$

This function can also be represented as a set of ordered pairs: $f = \{(a, x), (b, z), (c, y)\}$

The **domain** and **range** of a function are sets defined as:

- Dom $f = \{x | (x, y) \in f\}$
- Ran $f = \{y | (x, y) \in f\}$

The **codomain** of f is the set of all possible values you can map to. In our function the codomain would be the set M. Note that the range is a subset of the codomain.

In our above example, Dom $f = \{a, b, c\} = K$ and Ran $f = \{x, y, z\} = M$

The domain of the above function is K while the range and codomain are both M.

In each mapping, the element in the domain is called a **pre-image**, and it is uniquely mapped to its corresponding **image** in the co-domain.

Function f mentioned above is known as a bijective function. A function is **bijective** if and only if it is both injective and surjective.

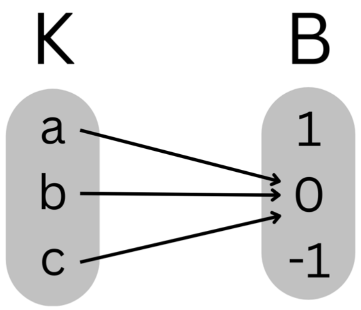

Figure 2: Many-to-one and Into

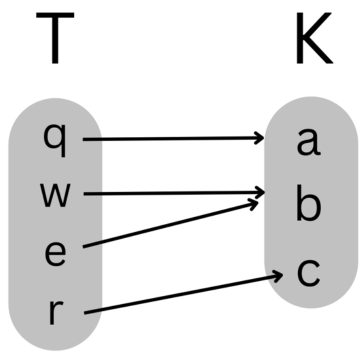

Figure 3: Many-to-one and Onto

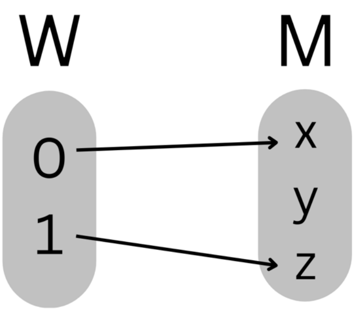

Figure 4: One-to-one and Into

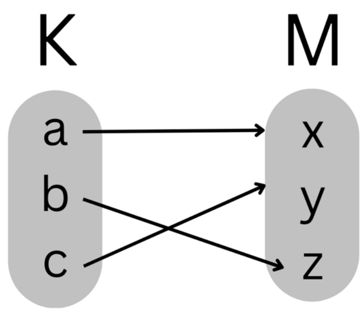

Figure 5: One-to-one and Onto

Figure 6: Different types of functions

A function is **injective** or **one-to-one** if and only if all elements of the domain are mapped to unique images in the codomain. In other words, no two pre-images share the same image. Conversely, a function that isn't one-to-one is a **many-to-one** function.

A function is **surjective** or **onto** if and only if its range is equal to its codomain. This means that every element in the codomain has a corresponding pre-image. A function that isn't surjective is an **into** function. Exmples of the different types of functions can be seen in Figure 6.

Having understood the concept of cardinality and the different types of functions, we can state the following:

- $|X| = |Y| \iff \exists$ a bijection between X and Y [1]

With the basics out of the way, we are now prepared to begin our exploration. To infinity and beyond!

4. Cantor's Theorem

What do you think the largest set is? Take a moment to mull that question over. Now, consider this: no matter how large a set you have in mind, I can present you with one that is even bigger. It's quite fascinating, isn't it? The power set of any set you can imagine will surpass its size. And if we continue, exploring the power set of that power set, the size expands endlessly. This prompts a question: Does the concept of the "largest" set exist? The answer is no, there does not exist a "largest" set. In the following section, we will prove this as a corollary to Cantor's theorem on the cardinality of power sets.

Theorem 1. *The power set of a set A has a cardinality greater than the cardinality of A. i.e.,*

$$|A| < |P(A)|$$

In other words, Cantor's theorem tells us that a power set always has more elements than the set itself.

To prove this theorem, we shall show, in two parts, that $|A|$ is neither greater than nor equal to $|P(A)|$. Thus, leaving us with only the possibility that $|A|$ is strictly lesser than $|P(A)|$.

Proof. Let us consider set A and its power set $P(A)$. First, we shall show that the cardinality of A is not greater than that of $P(A)$.

For every element $x \in A$, a corresponding element $\{x\}$ exists in $P(A)$. Thus, the cardinality of A cannot be greater than that of its power set.

Next, we need to show that the cardinality of A is not equal to that of $P(A)$. We are going to prove this by contradiction. So, let us assume that the cardinalities of A and $P(A)$ are equal. This would mean that there exists a bijection from A to $P(A)$. i.e.,

$$\exists f : A \xrightarrow{\sim} P(A)$$

In our mathematical universe, at the start of every space voyage, a captain must be democratically chosen by the crew. If we were to ask any crew member, they would have a list of candidates who they think should run for captaincy.

Let set A comprise all the crew members that voted in the elections. The subset related to any given person, say James for example, are those he thinks would make a better captain than most. Let this relation from the person to their list be the bijective function f. It is possible that James is a part of his own list. If this is the case, then let us call James (and others like him) "Vain". Alternatively, if they are not a part of their own list, let us call them "Humble".

To put this in mathematical terms we have,

$$V \text{ (for "Vain")} = \{x \in A \,|\, x \in f(x)\}$$
$$H \text{ (for "Humble")} = \{x \in A \,|\, x \notin f(x)\}$$

Now let us look at $H \subset A$, the set of all humble people in A. Given our assumption that f is bijective, there must be some person, say Tara, in A that believes only humble people (those in H) should be the captain.

Mathematically, given that f is a bijection, there exists some $m \in A$ such that $f(m) = H$.

Surely, Tara must be either humble or vain, but not both.

However, if Tara is humble, she must be in H. This means that she is on her own list of ideal candidates, which makes her vain.

If Tara is vain, she is not in H. But then she is not a part of her list, which makes her humble. i.e.,

$$t \in H \implies t \notin f(t) \implies t \notin H$$
$$t \notin H \implies t \in f(t) \implies t \in H$$

In both the cases, we get a contradiction. Hence, our only assumption that there exists a bijection from A to $P(A)$ must not be true.

This means that the cardinality of A is not equal to that of $P(A)$.

We have proved that the cardinality of A is neither greater nor equal to the cardinality of $P(A)$. Thus, we can conclude that the cardinality of $P(A)$ is strictly greater than the cardinality of A. □

Corollary 1. *There is no "largest" set.*

Let us prove this by contradiction. We call a set A "larger" than another set B if the cardinality of A is greater than that of B.

Let set K be the largest set. We know that there exists the power set of K, $P(K)$.

From our assumption that K is the largest set we get, $|K| > |P(K)|$.

This is a contradiction to the above-mentioned and proven Cantor's theorem. Hence, our only assumption that K is the largest set must be false.

Thus, there does not exist a set that is larger than all other sets.

5. Types of Infinity: Countable and Uncountable

Consider the following sets: $A = \{1, 2, 3\}$ and $B = \{p, q, r\}$. If we were asked to compare the sizes or cardinalities of these two sets, we could do so easily. We can count the elements in each set individually and compare the two. Here, $|A| = |B| = 3$. This method of counting the elements works well for smaller finite sets. The method, although cumbersome, is possible for larger finite sets as well. However, it isn't possible to count sets with an infinite number of elements. In this case, an alternative method is used. To conclude that two sets are of the same size, we need to be able to pair up all elements of the two sets. If we tried pairing up the elements of A and B in order, we would get the following ordered pairs: $(1, p), (2, q), (3, r)$. Since each element in A is paired up with an element in B, we can say that A and B are of equal size. Note that we have essentially shown that there exists a bijective function between A and B with our ordered pairs.

Now, let's look at a set with infinite cardinality. The set of natural numbers starts at 1, and each successive element increases by 1, continuing on until infinity.

$$\mathbb{N} = \{1, 2, 3, 4, 5 \ldots\}$$

Although this set has an infinite cardinality, we can list the elements if we are patient enough. This is possible because, between any two natural numbers, there is a finite number of natural numbers. There is a discrete nature to this set that makes it possible to count it. However, not all sets are listable this way. The set of all real numbers is uncountably infinite. This signifies that even the most resilient among us cannot list all the elements in this set. This is because the set encompasses not just infinitely larger numbers similar to the natural numbers, but also an infinite continuum of real numbers between any two distinct real numbers. The cardinality of this set surpasses that of the set of natural numbers. This means that despite both sets having an infinite number of elements, one set's infinity is larger than the others.

To gain a better understanding of this abstract mathematical concept, let's delve into a space analogy. Consider the following two scenarios: a single universe and a multiverse. Arya, the ambitious astrophysicist, wants to compile a comprehensive list of all existing stars. In the first scenario, which involves a single, boundless universe, there would be an infinite number of stars. However, Arya could start his list by focusing on the stars closest to him and gradually expand outward. As he reaches stars ten light years away, he would have an infinite number of stars left to list. Nevertheless, he can be certain that he has listed all stars within the ten-light-year radius that he has covered. This mirrors the concept of natural numbers—an infinite set, yet countable.

Now, envision Arya living in the second scenario: a multiverse. Here, the complexity deepens. Not only are there infinitely numerous stars within each universe, but there's also an infinite count of these boundless universes. However, that's not all. Between any two distinct stars, an infinite number of additional stars exists. This scenario parallels the intricacy encountered when grappling with the set of real numbers.

6. The Uncountable Real Numbers

Theorem 2. *The set of all reals is uncountably infinite. Mathematically,*

$$|\mathbb{R}| \neq |\mathbb{N}|$$

Proof. Let us consider the set $X = (0, 1)$. X is the set of all the real numbers between 0 and 1, excluding both 0 and 1 themselves. We can show that $\mathbb{R}$, the set of real numbers, is uncountably infinite by proving that X is uncountably infinite. This is because X is a proper subset of $\mathbb{R}$. We will prove that X is uncountable by contradiction.

Let us assume that X is a countably infinite set. i.e., $|X| = |\mathbb{N}|$. This means that there exists a bijection $f : \mathbb{N} \xrightarrow{\sim} X$. So, for each element in the set of natural numbers, n maps to a decimal number as shown below.

$$1 \mapsto 0.d_{1,1}d_{1,2}d_{1,3}d_{1,4}d_{1,5}...$$
$$2 \mapsto 0.d_{2,1}d_{2,2}d_{2,3}d_{2,4}d_{2,5}...$$
$$3 \mapsto 0.d_{3,1}d_{3,2}d_{3,3}d_{3,4}d_{3,5}...$$
$$4 \mapsto 0.d_{4,1}d_{4,2}d_{4,3}d_{4,4}d_{4,5}...$$
$$5 \mapsto 0.d_{5,1}d_{5,2}d_{5,3}d_{5,4}d_{5,5}...$$
$$\vdots$$

Since $\mathbb{N}$ is the "smallest" infinity, X must be "larger" than $\mathbb{N}$ if $|X|$ is not equal to $|\mathbb{N}|$. To prove this, we must show that the above function f is not a bijection. We can prove that by disproving the surjectivity of f. To show that f is not surjective we must find a decimal number in $\mathbb{R}$ that does not have a pre-image in $\mathbb{N}$. That is, we need to find a unique real number that is not in our above-shown list.

Let us build this unique number in steps, one for each natural number. In step 1, we will define the first decimal place, in step 2, we will define the second decimal place, and in step k, we will define the k^{th} decimal place.

Let us begin creating our unique real number. In step 1, $k = 1$, we define the first decimal place in the number. We will denote the number in the first decimal place by p_1. While picking a number in step 1, we need to make sure that $p_1 \neq d_{1,1}$. We do this to make sure that our final number is unique from the first number in our list. To do this, let

$$p_1 = d_{1,1} + 1, \text{ if } d_{1,1} \neq 9$$
$$p_1 = 0, \text{ if } d_{1,1} = 9$$

By these rules, we make sure that $p_1 \neq d_{1,1}$.

We continue to define p_i for all $k \in \mathbb{N}$ using the same rules as for p_1. i.e.,

$$p_k = d_{k,k} + 1, \text{ if } d_{k,k} \neq 9$$
$$p_k = 0, \text{ if } d_{k,k} = 9$$

Thus, we can be sure that $p_k \neq d_{k,k}$.

After completing all our steps, we will have a sequence $p_1, p_2, p_3, p_4, p_5, ...$ Using this sequence we can construct a new real number p.

$$p = 0.p_1p_2p_3p_4p_5...$$

This number is unique and not on our list. As if it were, then $f(k) = p$ for some $k \in \mathbb{N}$. However, the k^{th}decimal place of $f(k)$ is different than the k^{th} decimal place of p by the rules we set while defining p. Which means the numbers differ from each other.

Therefore, p is not on our list. Meaning that f is not a surjection. This is a contradiction to the fact that f is a bijection. Therefore, our only assumption that X is countable must be false.

Thus, we have that X and in turn, $\mathbb{R}$ is an uncountably infinite set. $\square$

7. Conclusion

In this exploration, we were able to uncover fascinating results about infinity guided by Georg Cantor. We explored the idea that there is no largest set and that even the most patient among us cannot count the set of real numbers. In closing, our exploration of infinity comes to an end. Just as astronauts return with new perspectives on our universe, I hope this expedition has ignited a newfound fascination with the boundless realm of mathematics.

References

[1] Jirí Lebl. *Basic analysis: Introduction to real analysis with University of Pittsburgh supplements*. Jirí Lebl, 2012, p. 19.

[2] Eli Maor and Ian Stewart. In: *To infinity and beyond: A cultural history of the infinite*. Princeton University Press, 2017, pp. 54–56.