Pittsburgh Interdisciplinary Mathematics Review
Volume 1, Number 1, April 2024, Pages 66–82
https://doi.org/10.5195/pimr.2024.22

Enough Model Theory to Understand Skolem's "Paradox"

David K. Philips

Department of Mathematics, University of Pittsburgh, Pittsburgh, PA
E-mail: dkp23@pitt.edu

1. Introduction

Usually in mathematics you learn a certain subject, and that subject deals with certain mathematical structures. If you take an algebra course you deal with groups, fields, and rings. If you take a graph theory course you deal with graphs. These are all examples of mathematical structures, and within these branches of mathematics, you learn about *specific* structures of which a given branch is focused on. Model Theory deals with the study of structures *in general*. In studying it we seek to create a sort of meta-analysis of mathematics as a whole. By virtue of being the "mathematics of mathematics," the subject can sometimes be technical. Accordingly, my goal with this article is fairly modest. That is, to give a foundation for further study. I will do this by discussing the basics, with an end reward being a theorem with some interesting consequences. Namely, the downward version of the Löwenheim–Skolem theorem.

2. Structures and Interpretations

2.1. Structures

If you are at all familiar with mathematics, you have come across a mathematical structure, even if you were not aware of it at the time. You can *loosely* think of structures as "the places where languages occur in." It should be noted that structures are not only a mathematical phenomenon, they can appear essentially anywhere with a consistent language. We will clarify the specifics of this later, but for now, here are some examples.

1. Given a field $\mathbb{F}$, a triple $\boldsymbol{V} = (V, +, \cdot)$, where "$V$" is a set and "$+$," "$\cdot$" are binary operations, $\boldsymbol{V}$is an $\mathbb{F}$-*vector space* if "$+$" and "$\cdot$" satisfy the following properties :
   (a) **Commutativity** $u + v = v + u \ \ \forall u, v \in V$
   (b) **Associativity** $(u + v) + w = u + (v + w)$ and $(ab)v = a(bv)$ $\forall u, v, w \in V$ and $\forall a, b \in \mathbb{F}$
   (c) **Additive Identity** $\exists\, 0 \in V$ such that $v + 0 = v \ \ \forall v \in V$

(d) **Additive Inverse** $\forall v \in V\ \exists w \in V$ such that $v + w = 0$

(e) **Multiplicative Identity** $1v = v\ \ \forall v \in V$

(f) **Distributive Properties** $a(u+v) = au + av$ and $(a+b)v = av + bv\ \forall a, b \in \mathbb{F}$ and $\forall u, v \in V$

2. A *partial ordering* is a pair $\boldsymbol{P} = (P, \leq)$, where "$P$" is a set and $\leq$ is a binary relation on P, satisfying the following properties :

(a) **Reflexivity** $\forall u \in P,\ u \leq u$

(b) **Antisymmetry** $\forall u, v \in P$ if $u \leq v$ and $v \leq u$ then $u = v$

(c) **Transitivity** $\forall u, v, w \in P$, if $u \leq v$ and $v \leq w$ then $u \leq w$

Let us now give a formal definition.

Definition 2.1. *A* structure *is a quadruple $\boldsymbol{S} = (S, C, F, R)$, where S is a set, C is a set of elements from S (constants), F is a set of operations on S (i.e. each element of F is a function from S^n to S for some $n \geq 1$). and R is a set of relations on S (i.e. each element of R is a subset of S^n for some $n \geq 1$)*

It should be noted that in [2] there are additional sets of relations that are required to make it into a structure. Also, we use "$\boldsymbol{S}$" to denote a structure and we use "S" to denote the underlying set of that same structure. Though this definition is adequate, it can lead to ambiguity. When we are comparing structures with one another, if we only had this definition, it would be difficult to talk about each structure distinctly. If one structure had a binary operation and you wanted to compare it with another structure that had another distinct binary operation, naturally, one would want specific names for each distinct operation in order to compare the two. This goes for operations, relations, and constants alike. We define these "names" by defining "signatures."

2.2. Signatures

Before I give the formal definition, here are some concrete examples of signatures.

Example 2.1. *The signature for graphs is*

$$\sigma_{gr} = (\emptyset, \emptyset, \{E\}, (E \mapsto 2)),$$

Though this would be the "official" signature of graphs, in order to make things cleaner we use the following notation:

$$\sigma_{gr} = (E),$$

and we make sure to note that "E" is a binary relation. Likewise, we use a similar methodology for any given signature.

Example 2.2. *The signature for arithmetic is*

$$\sigma_{arthm} = (0, S, +, \cdot),$$

where "0" is a constant symbol, "S" is a unary function symbol (the successor), and "+","·", are binary function symbols.

Example 2.3. *The signature for sets is*

$$\sigma_{set} = (\in),$$

where "∈" is a binary relation symbol.

Definition 2.2. *A* signature *is a quadruple* σ=(C,F,R,α)*, where C,F,R are pairwise disjoint sets (of symbols), which we refer to as the* sets of constant, function, *and* relation symbols *respectively, and*

$$\alpha : F \cup R \to \mathbb{N}$$

which we refer to as the arity function *and call* $P \in F \cup R$ *an* n-ary *symbol if* $\alpha(P) = n$*. We set* $\mathrm{Const}(\sigma) := C$*,* $\mathrm{Func}(\sigma) := F$*,* $\mathrm{Rel}(\sigma) := R$*. Furthermore, for each* $n \in \mathbb{N}$*, we denote* $\mathrm{Func}_n(\sigma)$ *and* $\mathrm{Rel}_n(\sigma)$ *to be the sets of n-ary symbols in* $\mathrm{Func}(\sigma)$ *and* $\mathrm{Rel}(\sigma)$*, respectively.*

Think of the arity function as a map between a function and its arity. For example, $f(x, y, z)$ would be a 3-ary function. As of now, the symbols within signatures are lifeless, as they do not yet represent elements. This is because there is no mechanism by which they are related. We should think of the sets within the signature purely as "names" for elements. In order to relate the sets of a signature to the actual elements themselves we must define the notion of a structure-signature pair, i.e. a "$\sigma - structure$."

2.3. σ-structures

Definition 2.3. *A* $\sigma - structure$ *is a pair* $\boldsymbol{S} = (S, i)$ *where S is a set and i is a map (correspondence) that assigns :*

- *to each* $c \in \mathrm{Const}(\sigma)$*, an element i(c) of S*
- *to each* $f \in \mathrm{Func}(\sigma)$ *an operation* $i(f) : S^{\alpha(f)} \to S$
- *to each* $r \in \mathrm{Rel}(\sigma)$*, a relation i(R)*$\subseteq S^{\alpha(R)}$

We call S the universe of the structure S.

This "interpretation function," i, essentially takes a symbol from the signature and maps it to its corresponding element within the structure. For simplicity, we write q^S instead of $i(q)$ for all symbols q in σ. So instead of (S, i), we write

$$\boldsymbol{S} = (S, (c^S)_{c\in C}, (R^S)_{r\in R}, (f^S)_{f\in F}).$$

A σ-structure, therefore gives an interpretation of each of the symbols in the signature. Again, it is only within a structure do they come to life. We could for example, in a given structure, use the symbol "1" to represent the constant element "2." Let us see an example of how this works in practice: The structure of the natural numbers is a σ_{arthm}-structure, we define it as follows: $\boldsymbol{N}= (\mathbb{N}, 0^N, S^N, +^N, \cdot^N)$. Here we define 0^N as our usual conception of the "0" symbol, S^N as a unary operation which adds "1" to any given constant, $+^N$ as our usual conception of the "+" symbol, and "$\cdot$" as our usual conception of the "$\cdot$" symbol. But we could, for example, define an alternate σ_{arthm}-structure (an unusual conception of the natural numbers), call it - N_{absurd}. We define N_{absurd} as follows : $\boldsymbol{N}_{absurd}= (\mathbb{N}, 0^{N_{absurd}}, S^{N_{absurd}}, +^{N_{absurd}}, \cdot^{N_{absurd}})$. Here we define $0^{\mathbb{N}_{absurd}}$ as "3", $S^{\mathbb{N}_{absurd}}$ as the operation which adds "1" to any given constant, $+^{\mathbb{N}_{absurd}}$ as our usual conception of the "+" symbol, and "$\cdot^{\mathbb{N}_{absurd}}$" as the function "$2x$." It should be fairly obvious N_{absurd} is not like our typical natural numbers. Nonetheless, it is a structure in the same signature. Going forward, if the interpretation is obvious, we will denote q^S simply as q, for any symbol q.

2.4. Substructures

Now we get to the notion of a substructure. We can think of them like structures within structures. Again as with structures, you have probably already seen many examples of substructures, subgraphs, subfields, and subgroups are a few examples. We will now give a formal definition.

Definition 2.4. *For σ-structures $\boldsymbol{A}, \boldsymbol{B}$ we say that $\boldsymbol{A}$ is a* substructure *of $\boldsymbol{B}$, written $\boldsymbol{A} \subseteq \boldsymbol{B}$, if $A \subseteq B$ and the interpretations of σ by A and B coincide on A, that is :*

- $c^A = c^B$*, for each* $c \in \mathrm{Const}(\sigma)$
- $f^A = f^B|_{A^{\alpha(f)}}$ *for each* $f \in \mathrm{Func}(\sigma)$
- $R^A = R^B \cap A^{\alpha(R)}$ *for each* $R \in \mathrm{Rel}(\sigma)$

To make this concrete, consider the structure, $(\mathbb{R}, 0, 1, +, -, \cdot)$. Some substructures of this structure would be $(\mathbb{N}, 0, 1, +, -, \cdot)$, $(\mathbb{Q}, 0, 1, +, -, \cdot)$, $(5\mathbb{R}, 0, 1, +, -, \cdot)$. Some non-examples would include $(\mathbb{C}, 0, 1, +, -, \cdot)$ and $(\mathbb{R}, 0, -,)$. For a σ-structure $\boldsymbol{B}$ and $A \subseteq B$, we say that A is a universe of a substructure of $\boldsymbol{B}$ if there is a substructure $\boldsymbol{A} \subseteq \boldsymbol{B}$ whose universe is A.

3. Morphisms

Definition 3.1. *A* morphism *is a structure-preserving map from one mathematical structure to another one of the same signature.*

The key takeaway from this is that we have a "structure-preserving map" from one structure to another. When we say "structure-preserving" it is like saying that the map preserves all the capabilities of one structure when applied to the other. For example, suppose $f : \boldsymbol{A} \to \boldsymbol{B}$ was a linear map (a structure-preserving map) between vector spaces $\boldsymbol{A}$ and $\boldsymbol{B}$. Since you can add any two vectors within $\boldsymbol{A}$ and get out another vector within $\boldsymbol{A}$, then the same thing happens in $\boldsymbol{B}$ after applying f. That is $f(a+b) = f(a)+f(b)\ \ \forall a, b \in A$. In the following subsections I would like to define three important morphisms. We start with homomorphisms, which are special morphisms, then move on to isomorphisms, which are special homomorphisms, and then finally automorphisms, which are special isomorphisms. In order to explain the differences between morphisms, we use an analogy. Say you have a language translation machine in which you input a sentence in one language and get out that sentence in another. Think of morphisms, as the machine itself and the specific types of morphisms as different versions of it.

3.1. Homomorphisms

Our first version of this machine is the homomorphism. It is good but not perfect. Say we turn the machine to Spanish mode and we take the sentences "El coche es rápido," which means "The car is fast," "El camión es rápido," which means "The truck is fast," and "El autobús es rápido," which means "The bus is fast," and we input them into the machine. The homomorphism machine would only return the English sentence "The car is fast." Notice how the machine does not have to correlate the objects of the sentence *perfectly*, but it does have to correlate them *similarly*. The word "Coche" means car so the machine can translate perfectly if it wishes, but "Camión" and "Autobús" are only just similar words so it does not have to. Additionally, notice how we do not have a one-one correspondence between sentences, each of these three sentences were sent to one sentence in the output. Just as in the analogy, real homomorphisms are structure-preserving maps from one structure to another, though they are not necessarily "one-to-one." Let us now make this all concrete.

Definition 3.2. *Let $\boldsymbol{A}, \boldsymbol{B}$ be σ-structures. A function $h : \boldsymbol{A} \to \boldsymbol{B}$ is called a σ-*homomorphism *(or just a* homomorphism*) if h respects the interpretation of σ, that is:*

- $h(c^A) = c^B$*, for each* $c \in \mathrm{Const}(\sigma)$
- $h(f^A(\vec{a})) = f^B(h(\vec{a}))$*, for each* $f \in \mathrm{Func}(\sigma)$ *and* $\vec{a} \in A^{\alpha(f)}$
- $R^A(\vec{a}) \Rightarrow R^B(h(\vec{a}))$*, for each* $R \in \mathrm{Rel}(\sigma)$ *and* $\vec{a} \in A^{\alpha(f)}$

Denote this by $h : \boldsymbol{A} \to \boldsymbol{B}$

We only require (⇒) for relations since requiring (⇐) to hold requires many homomorphisms to be injective.

3.2. Isomorphisms

Our next machine version is the "isomorphism." Say we again want to translate the Spanish sentence "El autobús es rápido." When we put our sentence through the isomorphism machine, it returns, only, the English sentence "The bus is fast." Unlike the homomorphism, the isomorphism translates the sentence perfectly. That is, the meaning of the proposition is exactly the same in both cases. We have in essence a "renaming" of a state of affairs. The state of affairs in our morphism world is that there is a car and it is fast, and in order to describe this we use languages. If we have a function that takes a sentence from one language and maps it to another sentence within another language and that sentence corresponds to the same state of affairs, then that function is an isomorphism. We now define this all formally.

Definition 3.3. *Let* $\boldsymbol{A}$,$\boldsymbol{B}$ *be* σ*-structures. A function* $h : \boldsymbol{A} \to \boldsymbol{B}$ *is called a* σ-isomorphism *(or just an* isomorphism*) if* h *is bijective and both* h *and* h^{-1} *are* σ*-homomorphisms; in this case we write* $h : \boldsymbol{A} \xrightarrow{\sim} \boldsymbol{B}$. *The structures* A, B *are called* isomorphic *if there is an isomorphism between them; denote this by* $\boldsymbol{A} \cong \boldsymbol{B}$.

3.3. Automorphisms

Lastly, returning to our analogy, we get the automorphism machine. If we take again the English sentence "The car is fast" and put it in our automorphism machine, we get the output "The automobile is fast." All it did was rename "car" into "automobile," even though both are terms equivalent. Also notice, how this time we are mapping the "English" structure back into itself, rather than extending the map into the "Spanish" structure. If we had a second, different, automorphism machine and we input the same sentence, we could also hypothetically get it back as an output. We require the second machine to be different since it is a key characteristic of automorphisms that each output has exactly one input.

Definition 3.4. *An* automorphism *is an isomorphism from a structure to itself.*

4. First Order Language

In order to move any further, we have to define the language in which we will be speaking. This language will allow us to express statements about structures. In order for this language to be relevant to us, it should mirror how we normally use language. With this in mind, the following definitions should seem fairly intuitive.

4.1. Alphabet

Definition 4.1. *The* alphabet $\mathbb{FOL}(\sigma)$ *of the first order language in the signature* σ *consists of the symbols in* σ *and the following additional symbols:*

- *logical symbols:* $\doteq \neg \wedge \vee \rightarrow \forall\ \exists$
- *punctuation symbols:* , ( )
- *variable symbols:* $v_0, v_1, v_2, \ldots$

The symbols $\forall$ *and* $\exists$ *are called* quantifiers*.*

Definition 4.2. *Finite sequences of symbols from* $\mathbb{FOL}(\sigma)$ *are called* words*.*

We use the symbol $\doteq$ instead of $=$. If we say $x_1 \doteq x_2$ we mean to say that these two objects are actually the same object. If we were to just write $v_1 = v_2$, it may be ambiguous in the sense that they may only be equal within a context. For example, if we were to say: "Marcus Aurelius" = "The Emperor of Rome," this would only be correct in the context of 161-180 CE, if we were to analyze that equivalence in the context of 27 BCE-14 CE it would be false, as Augustus would have been ruling. But, if we were to say: "Marcus Aurelius" $\doteq$ "Marcus Aurelius," this would be indicating that "Marcus Aurelius" is the exact same object as "Marcus Aurelius," and this would hold true regardless of any context. Additionally, we casually use letters different than $v_1, v_2, v_3, \ldots$ to denote variables. For example, "u, v, w, x, y, z," etc.

4.2. Terms

Notice we define "words" as "finite sequences of symbols," this definition does not require words to be intelligible. Under this definition, "$\exists \rightarrow \vee v_3()()$" and "$\vee \rightarrow v_3\forall$" can be considered words. We need a way to specify which types of words are meaningful and in order to do this, we define "terms."

Definition 4.3. *A* σ-term *(or* term *in* $\mathbb{FOL}(\sigma)$*) is a word formed via the following recursive rules:*

- *each* $c \in \mathrm{Const}(\sigma)$ *is a term;*
- *each variable is a term;*
- *if* $t_1, ..., t_n$ *are terms and* $f \in \mathrm{Func}_n(\sigma)$*, then* $f(t_1, ..., t_n)$ *is a term.*

We let "Terms(σ)" denote the set of all σ-terms.

Example 4.1. "$x + 0$" *is a term in* $\mathbb{FOL}(\sigma_{arthm})$*.*

Example 4.2. "0 " *is a term in* $\mathbb{FOL}(\sigma_{arthm})$*.*

Example 4.3. *"$xy + S(0)$" is a term in $\mathbb{FOL}(\sigma_{arthm})$.*

Non-Example 4.1. *"$\wedge\vee \rightarrow xw$" is not a term in $\mathbb{FOL}(\sigma)$.*

Technically, 4.1 should be written $+(x, 0)$ where "+" is the function that adds "x" and "0," and 4.3 should be written $-(\cdot(x, y), S(0))$ where "+" is the function that subtracts "xy" from $S(0)$, and "$\cdot$" is the function which multiples "x" and "y." But, we do not use this notation because it would make it unnecessarily difficult to write anything. Just as the symbols in the signature of a structure rely on the structure for meaning, terms do also. For example, if we had structures $\boldsymbol{A} = (\mathbb{N}, 1^A, +^A)$ where 1^A is the usual 1, $+^A$ is the usual +, and $\boldsymbol{B} = (\mathbb{N}, 1^B, +^B)$ where 1^B was 3 and $+^B$ was our usual "$\cdot$" then, letting $t := (v_1 + v_2)$, we have that $t^A[1, 1] = 2$ and $t^B[1, 1] = 9$. Now we arrive at a lemma that will help us with proofs later on.

Lemma 4.1. *Let $\boldsymbol{A}, \boldsymbol{B}$ be two σ-structures. If $h : \boldsymbol{A} \rightarrow \boldsymbol{B}$ is a homomorphism, then for any term $t(\vec{v})$ and $\vec{a} \in A^{|\vec{v}|}$*

$$h(t^A(\vec{a})) = t^B(h(\vec{a})).$$

Proof. Use induction on the construction (length) of t. □

4.3. Formulas

Now we arrive at the notion of a "formula." Formulas are the mechanism by which we can conjoin terms with one another and prescribe truth values to their conjunctions. Right now all we have are terms and having no formulas to relate these terms by is like knowing the statements "I blinked" and "I am able to blink" are true but not knowing that the statement "if I blink then I am to blink" is also true. Poincaré once said, "Mathematics is the art of giving the same name to different things." If this is the case then formulas are the paintbrushes we use when creating this art.

Definition 4.4. *A σ-formula (or* formula *in $\mathbb{FOL}(\sigma)$) is a word formed via the following recursive rules:*

- *if s,t are terms, then $s \doteq t$ is a formula;*
- *if $t_1, ..., t_n$ are terms and $R \in \mathrm{Rel}_n(\sigma)$, then $R(t_1, ..., t_n)$ is a formula;*
- *if ϕ and ψ are formulas, then $\neg(\phi)$,$(\phi) \wedge (\psi)$,$(\phi) \vee (\psi)$, $(\phi) \rightarrow (\psi)$ are formulas;*
- *if ϕ is a formula and v a variable symbol, then $\forall v\phi$ and $\exists v\phi$ are formulas.*

We let "Formulas(σ)" denote the set of all σ-formulas. The formulas in the first two bullets are called "*atomic*." If a formula is formed without quantifiers (bullet four), it is called "quantifier-free." Additionally, a variable that is not quantified is called "free". A formula without any free variables is called a *sentence*. In a given structure, we interpret a formula as a relation. As I mentioned before with the example of "I blinked" and "I am able to blink," we can think of the statement "if I blink then I am able to blink" as a relation between the two smaller statements. If instead of statements like "I blinked" and "I am able to blink," we replaced them with variables "x_1" and "x_2," we can consider the statement "if I blinked then I am able to blink" as an instance of the formula "if () then ()." We can then call this formula "ϕ," and since it can take two inputs, we can assign a vector of variables $\vec{X} = (x_1, x_2)$ to it and call it a binary relation. In total, we have $\phi[\vec{X}]=$ "if x_1 then x_2." We call ϕ an "extended formula." In general, if you have a formula with n possible inputs, call it an "n-ary" relation and assign it a vector of n many variables. Let us now give a concrete definition.

Definition 4.5. *Let ϕ be a formula and let $\vec{v}$ be a vector of variables of $\mathbb{FOL}(\sigma)$. We call the word $\phi[\vec{v}]$ an* extended σ-formula *if $\vec{v}$ includes all of the free variables of ϕ and does not contain any variable that is quantified in ϕ.*

We let "ExtFormulas(σ)" denote the set of all extended σ-formulas. For an extended formula $\phi(\vec{v})[\vec{a}]$, if the interpretation of it in a structure $\boldsymbol{A}$ is true, we say $\boldsymbol{A}$ satisfies (or models) $\phi(\vec{v})[a]$, written $\boldsymbol{A} \models \phi(\vec{v})[\vec{a}]$.

Definition 4.6. *The set of all true sentences within a structure $\boldsymbol{M}$ is denoted $(\boldsymbol{M})$. Formally, $(\boldsymbol{M}) := \{\phi :\ \phi$ is a sentence and $\boldsymbol{M} \models \phi\}$. We call $(\boldsymbol{M})$ the "theory of $\boldsymbol{M}$."*

Lemma 4.2. *Let $\boldsymbol{A}$, $\boldsymbol{B}$ be two σ-structures. If $h : \boldsymbol{A} \to \boldsymbol{B}$ is an isomorphism, then for any formula $\phi(\vec{v})$ and $\vec{a} \in A^{|\vec{v}|}$*

$$\boldsymbol{A} \models \phi(\vec{a}) \iff \boldsymbol{B} \models \phi(h(\vec{a})).$$

Proof. We use induction on the construction of first-order formulas ϕ. For the induction step, only consider:

$$\phi = \neg\psi,\ \phi = \psi_1 \wedge \psi_2,\ \phi = \exists v\psi$$

as all other formulas can be built from these. We are essentially trying to go through all possible formulas contained within a structure and verify that, for any possible kind of formula you can build using 4.4, the lemma holds. Given that 4.4 only defined a formula in first-order logic, we can only say this lemma holds within a first-order context. If we had a different system in which we constructed formulas differently, it would not immediately follow from this proof that the lemma holds within that system. If $\phi = (t_1 \doteq t_2)$, then:

$$\boldsymbol{A} \models \phi(a) \iff t_1^A(a) \doteq t_2^A(a) \iff h(t_1^A(a)) \doteq h(t_2^A(a))$$

$$\iff t_1^B(h(a)) \doteq t_2^B(h(a)) \iff \boldsymbol{B} \models \phi(h(a))$$

If $\phi = R(t_1, ..., t_n)$ then:

$$\boldsymbol{A} \models \phi(a) \iff R^A(t_1^A(v)(a), .., t_n^A(v)(a)) \iff R^B(h(t_1^A(v)(a)), .., h(t_n^A(v)(a))) \iff \boldsymbol{B} \models \phi(h(a))$$

If $\phi = \neg\psi$ then:

$$\boldsymbol{A} \models \phi(a) \iff \boldsymbol{A} \models \neg\psi(a) \iff \boldsymbol{A} \not\models \psi(a) \iff \boldsymbol{B} \not\models \psi(h(a)) \iff \boldsymbol{B} \models \phi(h(a))$$

If $\phi = \psi_1 \wedge \psi_2$ then:

$$\boldsymbol{A} \models \phi(a) \iff \boldsymbol{A} \models \psi_1(a) \wedge \boldsymbol{A} \models \psi_2(a) \iff \boldsymbol{B} \models \psi_1(h(a)) \wedge \boldsymbol{B} \models \psi_2(h(a)) \iff \boldsymbol{B} \models \phi(h(a))$$

If $\phi = \exists v\psi$ then:

$$\boldsymbol{A} \models \phi(a) \iff \exists a' \in A, \boldsymbol{A} \models \psi(\overrightarrow{a}, a') \iff \exists a' \in A, \boldsymbol{B} \models \psi(h(\overrightarrow{a}), h(a'))$$

$$\iff \exists b' \in B, \boldsymbol{B} \models \psi(h(\overrightarrow{a}), b') \iff \boldsymbol{B} \models \phi(h(a))$$

□

5. Elementarity

Elementarity describes when structures agree on first-order truths. However, this does not mean that they are the exact same structure, only that they are alike in first-order ways. It can be the case that we have two structures that agree on all first-order formulas but that they disagree on formulas outside the bounds of first-order logic.

5.1. Elementary Equivalence

Definition 5.1. *Let* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *be* σ*-structures. We say that* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *are* elementarily equivalent *written* $\boldsymbol{A} \equiv \boldsymbol{B}$*, if* $(\boldsymbol{A}) = (\boldsymbol{B})$.

By Lemma 4.3.3, we can see that isomorphic structures are elementary equivalent. But the converse does *not* hold. For example, if we take two elementary equivalent structures $(\mathbb{Q}, <)$ and $(\mathbb{R}, <)$ (a good exercise is to show that these two are elementary equivalent), we can see that they are not isomorphic. This is because of the countability of rationals and the uncountability of the reals.). Saying these two structures are isomorphic implies there exists a bijection between $\mathbb{N}$ and $\mathbb{R}$ which is false. It will be important later to know when substructures agree on formulas.

Proposition 5.1. Substructures agree on quantifier-free formulas. *That is to say that for* σ*-structures* $\boldsymbol{A} \subseteq \boldsymbol{B}$*, any quantifier-free* σ*-formula* ϕ*, and* $\vec{a} \in A^n$*, we have :*

$$\boldsymbol{A} \models \phi(\vec{a}) \iff \boldsymbol{B} \models \phi(\vec{a}).$$

Proof. Use induction on the construction of ϕ. It may be helpful to use Lemma 4.3.3. for the base case. □

5.2. Elementary Substructures

We now define a new type of substructure.

Definition 5.2. *Let $\boldsymbol{A}$,$\boldsymbol{B}$ be σ-structures. We call a structure $\boldsymbol{A} \subseteq \boldsymbol{B}$* elementary*, written $\boldsymbol{A} \preceq \boldsymbol{B}$, if the inclusion map $\boldsymbol{A} \subseteq \boldsymbol{B}$ preserves all first-order formulas.*

By the above definition, if $\boldsymbol{A} \preceq \boldsymbol{B}$ then $\boldsymbol{A} \equiv \boldsymbol{B}$. Note, that the converse does not hold. Consider $2\mathbb{Z}$ and $\mathbb{Z}$. These structures are elementary equivalent but consider, for example, the sentence "$\forall y\ \exists x$ such that $x + x = y$," this is false in both structures so it does not violate elementary equivalence. But there is no such y in $2\mathbb{Z}$ that this sentence holds and there is one in $\mathbb{Z}$ so these two structures cannot be elementary.

5.3. Tarski-Vaught Test

Given our new notion of elementary substructures, it raises the question of how we can tell if a certain substructure is elementary or not. Since elementarity is characteristically the notion of a smaller structure abiding by the same first-order formulas as a larger structure, in order to check if two structures are elementary, we would have to confirm that any formula satisfied in the larger structure is also satisfied in the smaller structure. To do this we would have to, again, use induction on the construction of ϕ. It suffices to only check the cases where ϕ is atomic, $\phi = \psi_1 \wedge \psi_2$, $\phi = \neg\psi$, and $\phi = \exists y\psi(\vec{x}, y)$ because all other cases can be built off these. But notice, by proposition 5.1.1, we already know that all formulas will translate over atomic formulas and since $\phi = \psi_1 \wedge \psi_2$, $\phi = \neg\psi$ are just straightforward applications of the inductive hypothesis, we are really just left with the question of when $\phi = \exists y\psi(\vec{x}, y)$. In order for an existential formula like this to be satisfied both in the larger structure and in the smaller structure, we would need a witness for it in the smaller structure. In other words, if we had a formula $\phi = \exists y\psi(\vec{x}, y)$ that held true in $\boldsymbol{B}$, and we wanted to check whether or not it true held in $\boldsymbol{A}$, we would have to find some $a \in A$, which would satisfy it. Only then could we say $\boldsymbol{A} \models \exists y\psi(\vec{x}, y)$ because we would know for certain there is a specific element within A that would satisfy it. With that intuition, we give the following proposition.

Proposition 5.2 (The Tarski-Vaught Test)**.** *Let $\boldsymbol{A}$ be a substructure of $\boldsymbol{B}$. $\boldsymbol{A}$ is an elementary substructure if and only if for every formula $\phi(\vec{x}, y)$ and $\vec{a} \in A^{|\vec{x}|}$,*

$$\boldsymbol{B} \models \exists y\phi(\vec{a}, y) \iff \exists a' \in A \text{ such that } \boldsymbol{B} \models \phi(a', \vec{a}).$$

Proof. For ($\Rightarrow$) Suppose $\boldsymbol{A} \preceq \boldsymbol{B}$ and verify if the Tarski-Vaught condition holds. For ($\Leftarrow$) Suppose the Tarski-Vaught condition holds and show by induction on the construction of ϕ that for every σ-formula $\boldsymbol{A} \models \phi(\vec{a}) \iff \boldsymbol{B} \models \phi(\vec{a})$. As we mentioned before only the "$\exists$" case should be non-trivial. □

If a set is a subset of another set, the map that takes the smaller set to itself within the larger set is called the inclusion map, i.e. $i(x) = x$.

6. Downward Löwenheim–Skolem

Now that we have established a basic foundation, we are ready to approach the "Downward Löwenheim–Skolem" theorem. This theorem essentially states that any satisfiable structure has a countable model. Maybe this does not appear at first to be that interesting but let me phrase it differently. Remember that when we have $\boldsymbol{A} \preceq \boldsymbol{B}$ we also get that $\boldsymbol{A} \equiv \boldsymbol{B}$. Now imagine if we considered an uncountable structure say $\mathbb{R}$, from our new theorem we know that there must be a countable elementary substructure of it, call it $\boldsymbol{P}$. This implies that the set of all true sentences in $\mathbb{R}$ is the same as in $\boldsymbol{P}$, even though they both have different cardinalities. It is almost as if we are saying we cannot "control" a set of sentences. Since, for example, if we were looking to construct a set of sentences that only apply to an uncountable model, it would be impossible to do so because they would also apply to some countable model. In addition, we can observe two structures that agree on all first-order sentences but have completely different cardinalities. Now that I have (hopefully) sparked your interest, how does one go about formulating a proof of this theorem? Our first step is defining one more preliminary notion, Skolem functions.

6.1. Skolem-Functions

Definition 6.1. *Let $\boldsymbol{B}$ be a σ-structure and $\phi(\vec{x}, y)$ be an extended σ-formula. A* Skolem function *for $\phi(\vec{x}, y)$ is a partial function $f_{\phi(\vec{x},y)} : B^{\vec{x}} \rightharpoonup B$ such that, for each $\vec{b} \in B^{(\vec{x})}$, if $\boldsymbol{B} \models \exists y \phi(\vec{b}, y)$, then $f_{\phi(\vec{x},y)}(\vec{b})$ witnesses this, i.e. $\vec{b} \in \operatorname{dom} f_{\phi(\vec{x},y)}$ and $\boldsymbol{B} \models \phi(\vec{b}, f_{\phi(\vec{x},y)}(\vec{b}))$.*

Skolem functions exist due to the Axiom of Choice so in general they are not definable. In the previous section we discussed finding witnesses (specific elements which make the formula true) for formulas of the form $\boldsymbol{B} \models \exists \phi(\vec{x}, y)$. Think of Skolem Functions as such witnesses. The function takes in formulas of the form $\exists \phi(\vec{x}, y)$ that are satisfied in $\boldsymbol{B}$ and outputs parameters that act as witnesses for them, i.e. $\phi(\vec{b}, f_{\phi(\vec{x},y)}(\vec{b}))$. It is worth noting that a formula may have multiple witnesses and the Axiom of Choice only makes one arbitrary selection of them. This does not change anything but keep in mind that most of the time it is not the case where we can describe a certain witness as *the* witness to a formula, rather just *a* witness for it.

6.2. Downward Lowenheim–Skolem

Theorem 6.1 (Downward Löwenheim–Skolem)**.** *Let $\boldsymbol{B}$ be a σ-structure and $S \subseteq B$. There exists $\boldsymbol{A} \preceq \boldsymbol{B}$ with $A \supseteq S$ such that $|A| \leq max(|S|, |\sigma|, |\aleph_0|)$.*

I will not give a full proof here, I only seek to give a general idea. In simple terms, this theorem tells us that given any structure with a subset, we can find an elementary substructure of it in which that elementary substructure is both contained within that subset and is countable. As expected, to prove this you may want to start with trying to construct a countable substructure given any arbitrary structure. Consider an arbitrary structure "$\boldsymbol{B}$." By the conditions of our theorem, we need to first ensure that B has a subset, but every structure contains the empty set so this condition is always satisfied. Now, starting from S, we want to recursively define our new countable elementary substructure as a union of "S_n's." Call our hypothetical, countable, structure "$\boldsymbol{A}$," so that $\boldsymbol{A} := \cup_{n \in \mathbb{N}} S_n$. The first step in our construction of $\boldsymbol{A}$ is adding every symbol within S into our signature. Since we assume $\boldsymbol{B}$ has a countable signature, by extension, S should also. So by adding all the symbols from S into our new signature, we again get a structure with a countable signature. Now in order for a substructure to be elementary, recall that the Tarkski-Vaught test tells us that we need to find witnesses in $\boldsymbol{A}$ for any formula of the form $\exists y \phi(\vec{a}, y)$ satisfied in $\boldsymbol{B}$. We construct each S_n to do just that. Given $S_0 := S$ is defined, for S_1 we use our Skolem functions on every single possible formula of the form $\exists y \phi(\vec{a}, y)$ that is satisfied in $\boldsymbol{B}$. By doing this we find witnesses for each such formula. Since the signature of $\boldsymbol{B}$ is countable, there is only a countable amount of such formulas. Since the Axiom of Choice only selects a single witness for each formula, we see that we have a countable amount of witnesses. We then append all of these witnesses to a new, countable, subset S_1. We then take the union of our initial subset S and S_1 and consider it a new structure, which is again countable. In this new structure, now that we have all these witnesses in it, we also need symbols to represent them. So we append all of the new witnesses' symbols to this new structure's signature. Since the initial structure's signature was countable and we have only added countably more symbols, we again have a countable signature. Just as before, all of these new witness symbols being added to our signature have increased the number of possible parameters for formulas of the form $\exists y \phi(\vec{a}, y)$. So we now need even more witnesses for these new parameters within the formulas. So again we use our Skolem functions and find witnesses and then append them to a subset S_2 and then append S_2 to $S \cup S_1$ to obtain $S \cup S_1 \cup S_2$. Now again, with the addition of S_2, we again have more constants added to our signature, so we are in need of more witnesses. So we utilize the same process to form S_3 and then S_4 and S_5 and so on. Now we can see why $\boldsymbol{A} := \cup_{n \in \mathbb{N}} S_n$ is precisely what we wanted. Since every layer is itself countable, by the Tarski-Vaught test, it is clear that $\boldsymbol{A}$ is a countable elementary substructure of $\boldsymbol{B}$.

6.3. Weak Downward Löwenheim–Skolem

Before I state the next theorem it would be helpful to give an idea of what a "theory" is. A *σ-theory* (or just theory) can be thought of as a set of σ-sentences. A theory$T$ is satisfiable if it has a model $\boldsymbol{M} \models T$, that is $\boldsymbol{M} \models \phi$ for each $\phi \in T$.

Theorem 6.2 (Weak Downward Löwenheim–Skolem)**.** *Any satisfiable theory has a countable model.*

Proof. Take a theory where $\boldsymbol{M} \models T$ and notice that $\emptyset \subseteq \boldsymbol{M}$. By the downward Löwenheim–Skolem theorem, we can say that $\boldsymbol{M}$ also has a countable elementary substructure, call it $\boldsymbol{A}$. Since $\boldsymbol{A} \preceq \boldsymbol{M}$, $(\boldsymbol{A}) = (\boldsymbol{M}) \subseteq T$. Thus, $\boldsymbol{A} \models T$. $\square$

7. Paradoxes

Now that we know that every satisfiable theory has at least one countable model we should realize that ZFC (set theory), the very thing that mathematics is built from, is a theory. This implies that ZFC also must have a countable model, call it $\boldsymbol{M*}$. But, remember that Cantor proved a theorem that states that $\mathbb{R}$ is uncountable, and he used ZFC to do so. Cantor's theorem implies that $\mathrm{Th}(\boldsymbol{M*})$ must include the sentence "there exists an uncountable set," which implies that $\boldsymbol{M*}$, a countable structure, contains within it an uncountable structure. Now if this were a true contradiction, either everything I have written in this article is false, or all of mathematics is. Now if the latter is true, we would all be in big trouble and if the former were true, I will not be making it into graduate school. Luckily, mathematicians today do not consider this seemingly paradoxical problem, to be a problem at all. Before we get to the solution of this paradox, we should note that it is not very surprising that a model would not capture the full essence of the theory for which it is a model. Here is a great analogy from the Stanford Encyclopedia of Philosophy - "*A mathematical model of a physical theory, for instance, may contain only real numbers and sets of real numbers, even though the theory itself concerns, say, subatomic particles and regions of space-time. Similarly, a tabletop model of the solar system will get some things right about the solar system while getting other things quite wrong. So, for instance, it may get the relative sizes of the planets right while getting their absolute sizes (or even their proportional sizes) wrong; or it may be right about the fact that the planets move around the sun, while being wrong about the mechanism of this motion (e.g., the planets don't really move around the sun because some demonstrator turns a crank!)*"[1]. Now, getting back to the actual solution, again consider our countable elementary substructure $\boldsymbol{M*}$. We have established that even though $\boldsymbol{M*}$ is countable, it must satisfy the sentence "there exists an uncountable set," namely $\mathbb{R}$. Now this time, instead of discarding all of mathematics, notice that $\boldsymbol{M*}$ modeling the sentence "there exists an uncountable set," just means that $\mathbb{R}^{M*}$ *appears* uncountable within $\boldsymbol{M*}$. Think about what it means to be countable. If a set S is countable, by definition, it means there exists some $f : \mathbb{N} \to S$ somewhere within the set-theoretic universe, where f is bijective. When we say S is uncountable we just mean to say there is no such f in the model of mathematics we work in. But, when we say that $\mathbb{R}^{M*}$ is uncountable we are not quantifying over the entire set-theoretic universe. We are just making a statement within $\boldsymbol{M*}$, particularly we are saying there is no bijective $f : \mathbb{R}^{M*} \to \mathbb{N}$ within $\boldsymbol{M*}$, thus $\mathbb{R}^{M*}$ appears uncountable to $\boldsymbol{M*}$. This does not mean that there is no such f outside of $\boldsymbol{M*}$ and it does not make $\mathbb{R}^{M*}$ uncountable in any sense that would cause a contradiction. Additionally, we know such a bijection does not exist within $\boldsymbol{M*}$ since $\boldsymbol{M*}$ satisfies the sentence that $\mathbb{R}^{M*}$ is uncountable. So it must be the case that another bijection exists somewhere outside of $\boldsymbol{M*}$ since we "see" $\boldsymbol{M*}$ as countable. Diverging from this argument, remember symbols only have meaning within the context of a structure. In a certain structure "1" could be defined as what our normal conception of "2" is and likewise with any other symbol, including "$\in^{M*}$." So another argument could be that the statement "$\mathbb{R} \in^{M*} \boldsymbol{M*}$," is just a statement about the symbol "$\in^{M*}$" and since we do not know the interpretation of this symbol within $\boldsymbol{M*}$, we cannot say anything meaningful about statements using "$\in^{M*}$."

8. Philosophy

If you have taken an introductory real analysis course you may immediately be off put by the previous section. You may wonder how we can have the notion of a version of $\mathbb{R}$ being countable even though it has been proven to be uncountable. Formally, you are appealing to the "uniqueness theorem" of $\mathbb{R}$ which states :

Theorem 8.1 (Uniqueness). *If $\mathbb{R}_X$ and $\mathbb{R}_Y$ are two systems satisfying the axioms of the real numbers, then $\mathbb{R}_X$ and $\mathbb{R}_Y$ are isomorphic.*

You would probably say that this theorem would imply that our countable $\mathbb{R}^{M*}$ must be isomorphic to the uncountable $\mathbb{R}$, which would be impossible. Fortunately, all of these worries are settled when we realize that $\mathbb{R}^{M*}$ does *not* satisfy all the axioms of the real numbers and so the theorem does not even apply in the first place. You may counter by asking how we can even consider $\mathbb{R}^{M*}$ to be a version of the real numbers if it does not satisfy all of its axioms. The key point to realize here is that $\mathbb{R}^{M*}$ was created as a consequence of the Downward Löwenheim–Skolem theorem (Theorem 6.2), which is a theorem of first-order logic. Being a first-order theorem, its truth within the first-order system does not necessarily translate into its truth within higher-order logical systems. With this being said, the theory 4.3 of real numbers, in its entirety, includes sentences of second-order logic not just first. First-order logic only allows quantification over variables whereas second-order logic allows quantification over subsets. Naturally, the types of sentences we can form using only first-order logic are different than the ones we can form using second-order logic. When we consider the theory of the reals within the first-order context, we can only consider formulas that are possible to construct using the first-order system (Section 4) as the constituents of this theory. The models of this theory need not satisfy the same sentences as the second-order theory of the reals, only ones that can be expressed in first-order language. This restriction of expression radically changes how the models of the theory of the reals appear (viewed within a first-order framework). To make this concrete we can take a look at a specific axiom of the reals which is not expressible within first-order logic.

Theorem 8.2 (Completeness of $\mathbb{R}$). *Every non-empty subset $A \subseteq \mathbb{R}$, that is bounded above, has a least upper bound.*

This can be formally written as:

$$\forall S, ((\exists x, S(x) \wedge (\exists y \forall x, S(x) \implies x \leq y)) \implies$$

$$((\exists a \forall x, S(x) \implies x \leq a) \wedge (\forall b, b < a \implies \exists x, S(x) \wedge b < x)).$$

The unary predicate "S" in the above formula represents the subset of the reals being quantified over. Since we are quantifying over subsets, we are now working in second-order logic rather than first. Again, the theory of the reals in first-order logic, is the same theory of the reals in second-order logic but we exclude all the sentences not expressible via the first-order language. This means we can have models of the reals which do not satisfy all the same sentences as our typical conception of the reals and these models may, in turn, look very different. Going back to $\mathbb{R}^{M*}$ we can see that this is a first-order model of the reals in which a second-order sentence is not fulfilled, so it naturally has much different properties than our typical conception of the real numbers, one of which being that it is countable.

Again going back to the question of why would we allow something like $\mathbb{R}^{M*}$ to be a legitimate model of the real numbers, you may ask, "What is the use in defining the real numbers as something that we cannot treat like real numbers?" The answer deals with the weakness of the first-order logic system rather than eagerness about "non-standard" models. Rather than being a paradox, Skolem himself recognized the consequences of his theorem as just a flaw of the first-order system. Though there are many benefits to the first order system and lots of interesting work being done in non-standard analysis, it is not really that we want $\mathbb{R}^{M*}$ to exist. More so its existence is just a consequence of the axioms we assumed. Whenever we create a formal system of logic, it is bounded by our construction of it. Creating a formal logic system requires us to assume axioms and axioms are only "true" apriori. With this in mind, it should not be too much of a surprise that any system founded upon axioms does not model reality perfectly. If it did, it would essentially require us to already know all the axioms we need to assume innately. Even if we could somehow create a perfect formal logic system founded upon axioms, assuming it is consistent, Gödel's "Incompleteness Theorem" tells us that it would not be able to prove its own consistency. This is all to say that every formal system is flawed in its own way and the system of first-order logic is no exception. So though $\mathbb{R}^{M*}$ is a legitimate first-order construction, its legitimacy undermines the value of being a first-order construction.

Acknowledgments

In writing this I *frequently* referenced Professor Anush Tserunyan's notes on Mathematical Logic [2], and most of the definitions you find here are directly from there. Additionally, I would also like to thank Professor Thomas Gilton, the person who taught the majority of this material, and the Painter family for making this possible.

References

[1] Timothy Bays. *Skolem's paradox*. 2014. URL: https://plato.stanford.edu/entries/paradox-Skolem/#::text=Skolem%E2%80%99.

[2] Anush Tserunyan. *Mathematical Logic*. 2024. URL: https://www.math.mcgill.ca/atserunyan/Teaching_notes/logic_lectures.pdf.