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A Review of Alfred North Whitehead's "Introduction to Mathematics"

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1. An Introduction to Mathematics

In 1911, Alfred North Whitehead published a short book *Introduction to Mathematics* (IM) intended for students wanting an explanation of the fundamental ideas of mathematics [19].

Whitehead's IM has enduring value because it was written not long after he and Bertrand Russell published their monumental three-volume work *Principia Mathematica* (PM) – a publication of immense historical significance for mathematics [11]. IM sheds light on Whitehead's view of mathematics at that time. For Russell, the PM was the fulfillment of the logicist thesis: the claim that all of mathematics is logic and nothing but logic. However, Russell might have claimed more than PM could deliver, because the PM included the axiom of choice and the axiom of infinity, which are not part of logic. The primary components of their system were predicate logic, relations, and types.

Russell and Whitehead both turned away from mathematics after becoming exhausted by the ten years they spent writing the PM. Russell wrote in his autobiography, "My intellect never quite recovered from the strain. I have been ever since definitely less capable of dealing with difficult abstractions than I was before. This is part, though by no means the whole, of the reason for the change in the nature of my work." In 1910, Whitehead resigned his Senior Lectureship at Trinity College, Cambridge, and moved to London, where he was unemployed for a year [7]. His interests shifted from mathematics to education, and IM was his first book after PM. IM is a thus a retrospective and transitional work, written for the masses by a man recovering from a long intellectual ordeal.

The book IM was published in the series "The Home University Library of Modern Knowledge," which published concise introductions on a broad range of topics for a general audience. We might compare this series with similar book series published today, such as the "Very Short Introduction," "For Dummies," and the French "Que sais-je?" Bertrand Russell also published two books in the Home University series, including his *The Problems of Philosophy* [10]. (As an aside, Russell's *Problems* was the topic of one of my college entrance essays when I was a young man.)

Although a polymath, Whitehead was a mathematical physicist above all, and his book IM contains a mixture of math and physics. The closest parallel to Whitehead today might be Roger Penrose, whose books cover a broad range of math, physics, and philosophy [8].

IM tells the famous story of Archimedes' eureka moment of devising a test of the purity of the gold in the king's crown by submerging it in water to measure its specific gravity. According to Whitehead, "This day, if we knew which it was, ought to be celebrated as the birthday of mathematical physics." IM was written in the early days of quantum physics and relativity, but Whitehead does not mention the new physics. Instead, IM recounts the development of classical physics, including the fundamental forces of gravity, electricity, and magnetism, and Newton's laws of motion.

Vectors and linear algebra were not nearly so well known by math and engineering students in 1910 as they are today. Whitehead's IM helped to popularize vectors as a tool of applied mathematics, especially in his presentation of topics such as the parallelogram rule for the addition of forces. Whitehead's earlier treatise on universal algebra contains several chapters on vectors, matrices, and forces. The treatise developed Cayley's theory of matrices and the properties of vectors as conceived by Grassmann and Gibbs. According to Whitehead in IM, "The idea of the 'vector', that is, of a directed magnitude, is the root-idea of physical science.... Thus, when in analytical geometry the ideas of the 'origin', of 'co-ordinates', and of 'vectors' are introduced, we are studying the abstracts conceptions which correspond to the fundamental facts of the physical world." Vectors are used to illustrate the main object of the book IM, which is to show students why mathematics "is necessarily the foundation of exact thought as applied to natural phenomena."

In Whitehead's technical writings, his meaning is sometimes obscure, because of his insistence on generality. However, in his books written for a large audience such as IM, there are many memorable sentences. One of the most widely quoted sentences from IM foretold the rise of automation: "Civilization advances by extending the number of important operations which we can perform without thinking about them." Another passage foreshadows the concepts of entropy and compression in information theory: symbolism "should be concise,... Now we cannot place symbols more concisely than by placing them in immediate juxtaposition. In a good symbolism therefore, the juxtaposition of important symbols should have an important meaning." Whitehead saw the quest for the permanent and the general as fundamental, "To see what is general in what is particular and what is permanent in what is transitory is the aim of scientific thought.... This possibility of disentangling the most complex evanescent circumstances into various examples of permanent laws is the controlling idea of modern thought."

In IM, Whitehead tells the well-known story that Alexander the Great impatiently asked for his geometry tutor to make the proofs shorter. His tutor Menaechmus made the famous reply, "In the country there are private and even royal roads, but in geometry there is only one road for all." Whitehead disagreed, "But if Menaechmus thought that his proofs could not be shorteneed, he was grievously mistaken,... Nothing illustrates better the gain in power which is obtained by the introduction of relevant ideas into a science than to observe the progressive shortening of proofs which accompanies the growth of richness in idea.... There are royal roads in science; but those who first tread them are men of genius and not kings."

2. Some and Any

Whitehead wrote in IM, "Mathematics as a science commenced when first someone, probably a Greek, proved propositions about *any* things or about *some* things, without specification of definite particular things. When Whitehead mentions the words *any* and *some*, he is referring to what is now called predicate logic: the logic of the quantifiers *for all* ($\forall$) and *there exists* ($\exists$). IM thus places proofs in predicate logic as the mythical starting point of mathematics.

This statement in IM is particularly remarkable, because Whitehead himself was slow to understand the significance of predicate logic. Russell learned of symbolic notation for quantifiers from Peano and Frege, who had a major influence on Russell starting in 1900, when Russell met Peano in Paris at the International Congress of Mathematicians. It was Peano who introduced the symbol $\exists$ for the existential quantifier, which was adopted in PM. Russell and Whitehead used (x) to mean "for all x," and it was only later that Hilbert introduced the symbol ($\forall$).

Surprisingly, it appears that Whitehead failed to grasp the significance of symbolic notation for quantifiers when he wrote *Treatise on Universal Algebra* (UA), which was published in 1898 [14]. UA developed algebra and geometry from axioms with precise definitions, very much in the spirit of Bourbaki a few decades later. The treatise is divided into books, and Book II is *The Algebra of Symbolic Logic*. Within that book, Whitehead has an entire chapter on *existential expressions*. From his citations in Book II, it is known that Whitehead was well aware of the research on logic by Charles Saunders Pierce and his students. Pierce had developed notation for quantifiers by 1883 [3]. Whitehead also cited Schröder's work, which introduced Pierce's ideas to Europe. However, it seems that Whitehead truly failed to grasp the significance of authors he cited in UA until after Russell's encounter with Peano in 1900.

Whitehead endorsed the logicist thesis by writing, "A notable discovery has been made by the joint and partially independent work of three men: Frege, Peano and Bertrand Russell –a German, an Italian and an Englishman–... of the generalized conception of the variable and of its essential presence in all mathematical reasoning. This discovery empties mathematics of everything but its logic" [18]. In this context, when Whitehead speaks of the variable, he means the bound variable in logic. He explained explicitly in

IM, "The idea of the undetermined 'variable' as occurring in the use of 'some' or 'any' is the really important one in mathematics." Whitehead admitted in IM that a full understanding of quantifiers had come only recently, "It was not till within the last few years that it has been realized how fundamental *any* and *some* are to the very nature of mathematics."

3. Relations and Axioms of Geometry

As mentioned above, one of the main components of the logic of PM was a theory of relations or predicates. In logic, a predicate is interpreted as a relation of zero, one, or more variables. A function is a special kind of relation. Over a period of many years, Whitehead developed various axiomatic systems of geometry (and by extension, the natural world). Each axiomatic system can be described as a collection of entities that are joined by relations that are governed by axioms. If we try to summarize Whitehead's life endeavor in a single sentence, we might turn to a sentence in IM: "The events of this ever-shifting world are but examples of a few general connexions or relations called laws."

Whitehead's investigation of geometry should be viewed within its historical context of the systematization of what has been called "the golden age of geometry, the hundred years between 1790 and 1890" [9, p. vii]. Jürgen Richter-Gebert's book on *old geometry* is recommended. In the theory of Cayley and Klein, the metric of each of the classical geometries (Euclidean, hyperbolic, elliptic geometry, and even Minkowski geometry) is constructed in a uniform way from the cross-ratio and a choice of conic in projective space. In the words of Cayley, "Projective geometry is all geometry," because it subsumes the others. A useful resource is Jeremy Gray's history [4]. Whitehead himself recommended the two volumes on projective geometry by Veblen and Young [21].

Whitehead was part of the axiomatization movement in mathematics that became widespread through Hilbert's influence. In the ancient and medieval world, the axiomatic reasoning of Euclid's *Elements* served as the paradigm. By the nineteenth century, logic and mathematical rigor had advanced far beyond Euclid's ancient standards. Mathematicians such as Pasch and Peano started to redo Euclid according to more rigorous standards. In 1899, Hilbert published *Grundlagen der Geometrie (Foundations of Geometry)*, which brought Euclid into the twentieth century with a fresh set of definitions and axioms [5].

According to Eves, "By developing a postulate set for Euclidean geometry that does not depart too greatly in spirit from Euclid's own, and by employing a minimum of symbolism, Hilbert succeeded in convincing mathematicians to a far greater extent than had Pasch and Peano, of the purely hypothetico-deductive nature of geometry. But the influence of Hilbert's work went far beyond this, for, backed by the author's great mathematical authority, it firmly implanted the postulational method, not only in the field of geometry, but also in essentially every other branch of mathematics. The stimulus to the development of the foundations of mathematics provided by Hilbert's little book is difficult to overestimate" [2].

Eventually, general foundations for mathematics were provided by the type theory of PM (and also, independently by the Zermelo axioms of set theory). These foundational systems have had an enormous influence on the growth of mathematics. Before then, each branch of mathematics was separately axiomatized. To axiomatize a domain of mathematics means to systematize it by deciding which relations are to be considered as fundamental and by enunciating a set of axioms from which the entire theory can be deduced.

During the decade that he and Russell devoted to the PM, as a participant in the axiomatization movement, Whitehead wrote books on the axioms of geometry and physics [17][15]. Whitehead's aim in one publication during that period was to generalize the axioms of static geometry to give a system of axioms that describe the dynamical laws of physics [16]. Each system in that book gives relations on a collection of undefined entities, subject to a list of axioms.

Even his earlier treatise on universal algebra was a project in axiomatization of abstract algebra. For example, the treatise contains the axioms for a general nonassociative semiring. Theorems were derived from the definitions and axioms in a style of abstract algebra that was brought into the mainstream of mathematics decades later by B. L. van der Waerden in his book *Moderne Algebra* [13].

4. Whitehead's Three Tests of Accuracy

How is an axiomatic mathematical theory to be judged? Whitehead suggested three tests of accuracy: *"The tests of accuracy are logical coherence, adequacy, and exemplification"*[20].[1] These words were written in 1926, several years before Gödel published his incompletness theorems, but Whitehead's three tests of accuracy might be understood more fully in a post-Gödelian context. (Recall that Gödel's first version of the incompleteness theorem was written about the mathematical system developed by Russell and Whitehead.) The first test of accuracy is logical coherence, which might be taken to mean syntactic consistency. According to Whitehead, only this first test is required to establish validity of reasoning [14, p. vi]. Adequacy is a pragmatic test. In a mathematical system, adequacy might mean that it contains sufficient arithmetic to carry out the constructions of Gödel's incompleteness theorem. In a general metaphysical system, adequacy might mean adequacy for the purpose of sustaining an advanced scientific civilization. Exemplification means that the theory admits a model (in the sense of model theory). The first and third tests, logical coherence and exemplification, are related by Henkin's theorem (1949): if a theory is syntactically consistent, then it has a model.

[1] Whitehead proposed these tests to evaluate metaphysical theories, but here the tests are applied to mathematical theories.

To conclude this review, we place IM in the larger context of Whitehead's other works. Following an earlier suggestion of Dorothy Emmet, Norman Sieroka argues that "Whitehead can only be understood properly if viewed through his earlier works on mathematics and the philosophy of science. Unfortunately, however, the popular philosophical understanding tends to divide Whitehead into three (something not even Heidegger or Wittgenstein managed)" [12]. This review is the first step of a larger project of mine to reassess the mathematics of Whitehead, as a key to understanding his later work. Victor Lowe recalled, "Looking backward thirty years later, when he [Whitehead] was a famous philosopher at Harvard, he told me that insofar as he could lay claim to any originality, he thought *On Mathematical Concepts of the Material World* was the most original thing he had done" [7, v.1, p. 296]. Yet, this work of Whitehead had never been fully assessed [6]. A good starting point for the mathematics of Whitehead is the special issue of *Logique et Analyse* on Whitehead's early work [1]. As suggested by this short review, Whitehead developed themes in IM that have enduring value.

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