Pittsburgh Interdisciplinary Mathematics Review
Volume 1, Number 1, April 2024, Pages 128–142
https://doi.org/10.5195/pimr.2024.26

Beach Math
Problems and Solutions

Stephen Arndt* **and Ryder Pham**†

Department of Computer Science, University of Pittsburgh, Pittsburgh, PA
**E-mail*: stephenda313@gmail.com
†*Email*: rtp26@pitt.edu

1. What is Beach Math?

You're on the beach. Sun on your skin, feet in the sand, waves crashing in the distance. Your mind begins to wander, and it wanders towards a math problem you once saw many weeks ago, but never had the time to think about. You *want* to understand the solution, but not because you have to. Beach Math represents these kinds of problems. Problems done for fun, and problems which show that math is not just useful, but *enjoyable*!

Disclaimer: The difficulty of these problems is subjective, and they are meant to be challenging for the average reader. Do not be discouraged!

2. Problems

2.1. Calm Waves (Easy)

Compute $\sqrt{2+\sqrt{2+\sqrt{\dots}}}$.

Calm Waves Extension 1

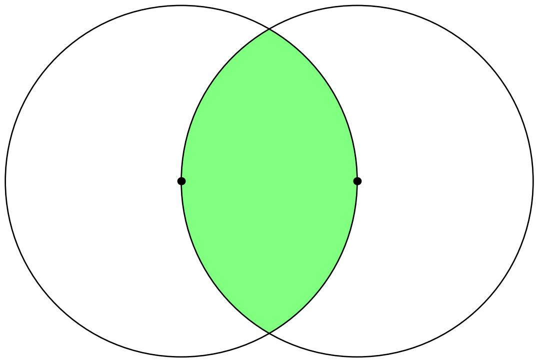
Compute $1+\frac{1}{1+\frac{1}{1+\frac{1}{\dots}}}$.

Calm Waves Extension 2

(Proposed by Ojas Mishra). Prove $\sqrt{2+\sqrt{2+\sqrt{\dots}}} = 2$ rigorously.

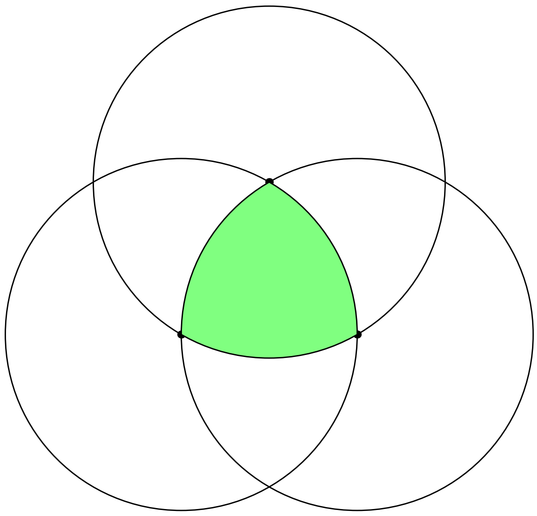
2.2. Rough Waters (Medium)

Suppose two circles of radius 1 are placed such that each circle passes through the center of the other circle. What is the area of the overlapping region?



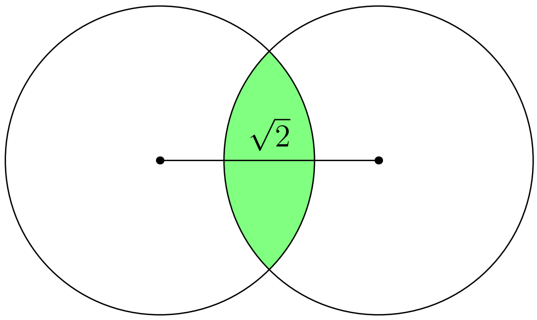
Rough Waters Extension 1

Suppose three circles of radius 1 are placed such that each circle passes through the center of the other two circles. What is the area of the overlapping region?



Rough Waters Extension 2

Suppose two circles of radius 1 are placed such that the circle's centers are distance $\sqrt{2}$ apart. What is the area of the overlapping region?



2.3. Tsunami (Hard)

(1989 Putnam A4). Suppose I pick a number α between 0 and 1. Can you always devise a game between us, using a fair coin, where the probability of me winning is α?

Tsunami Extension 1

What if the coin is unfair, i.e. flips on one side with probability p and the other side with probability $1-p$ for $p \neq \frac{1}{2}$? How can we answer the same original question using such a coin?

Tsunami Extension 2

What if you want to decide a winner among a group of multiple people using a fair coin, where each person has some probability of winning p_i? What about an unfair coin?

Tsunami Extension 3

On average, how many flips should our final "binary expansion" game take using a fair coin? What about the other games from Extension 1 and Extension 2?

3. Solutions

Calm Waves

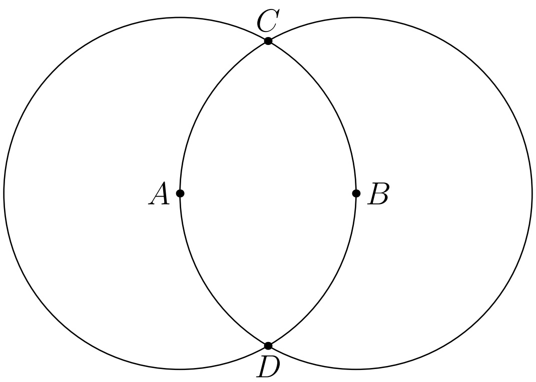
Let $x = \sqrt{2+\sqrt{2+\sqrt{\dots}}}$. Then $x = \sqrt{2+x}$. Square both sides to give $x^2 = 2+x$, and rearrange to give $x^2 - x - 2 = 0$. Factoring, we obtain $(x-2)(x+1) = 0$, giving the solutions $x = 2$ and $x = -1$. Of course, $x = -1$ is an extraneous solution, giving $x = 2$ as the correct solution. Thus,

$$\boxed{\sqrt{2+\sqrt{2+\sqrt{\dots}}} = 2}$$

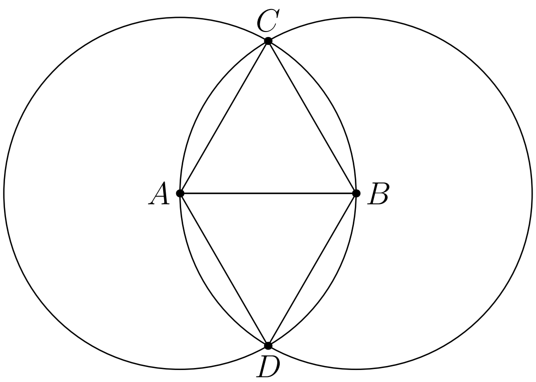
Rough Waters

The image shown is a classic Venn Diagram, and this problem asks us to compute the area of the central region shared by both circles. However, this region is a bizarre one, and we don't immediately know a formula for computing its area. How should we proceed? Commonly, when one runs into a problem they have not encountered, they should proceed by **breaking down** the problem into **subproblems!** If we can break this region down into parts, where we can compute the area of each part, we will be able to compute the area of the whole region.

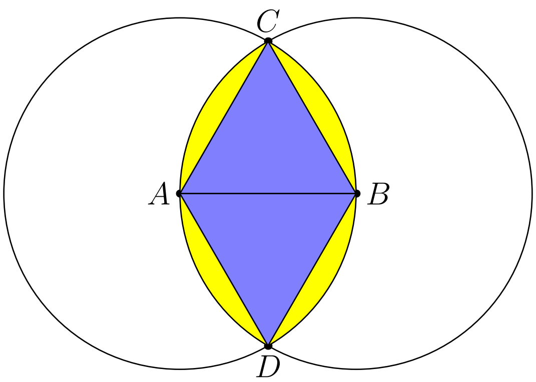
There are four important points on the boundary of this region.



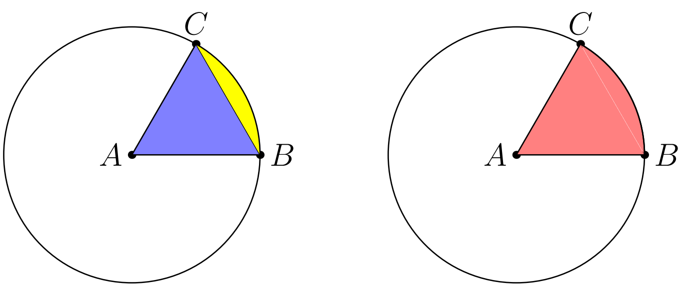
Let's connect them with lines to split the region up.



We can see that the region is comprised of 2 triangles and 4 "bananas".



We know how to compute areas of the triangles, but what about the bananas? The key insight is that each banana is just a sector minus a triangle.

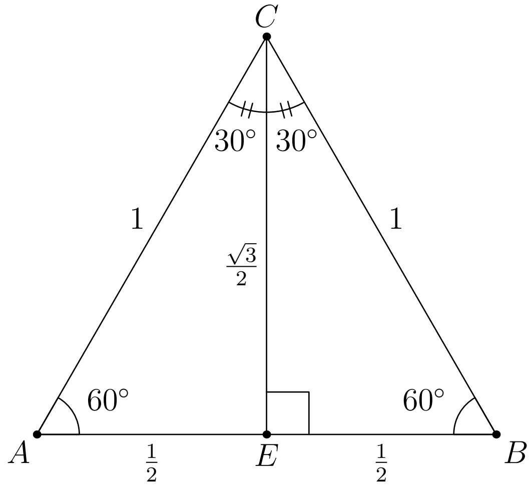


Let's clean things up algebraically. Let A be the area of the whole region, T be the area of the triangle, B be the area of the banana, and S be the area of the sector. Thus, we have

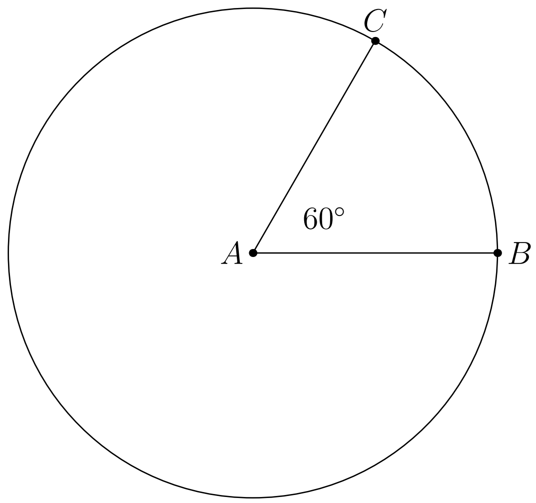
$$A = 2T + 4B = 2T + 4(S - T) = 4S - 2T$$

$$A = 4S - 2T$$

We are left with computing S and T. First, let's compute T. Note each triangle is an equilateral triangle of side length 1. Using some geometry, we obtain the below picture.



This gives $T = \frac{\sqrt{3}}{4}$. Note the height $h = |\overline{CE}|$ can be obtained by using Pythagorean Theorem, or by recalling the formulas for the sides of $30-60-90$ triangles. Because the triangles are equilateral, the sectors have central angle $60°$, and thus comprise $\frac{1}{6}$ of the total area of each circle.



Because each circle has area $\pi r^2 = \pi$, each sector has area $\frac{\pi}{6}$. Putting it together, we have

$$
\begin{aligned}
A &= 4S - 2T \\
&= 4\left(\frac{\pi}{6}\right) - 2\left(\frac{\sqrt{3}}{4}\right) \\
&= \boxed{\frac{2\pi}{3} - \frac{\sqrt{3}}{2}}
\end{aligned}
$$

which constitutes $\boxed{\approx 39.1\%}$ of each circle's area. Quickly checking this result visually, we see that our answer is reasonable.

Tsunami

3.0.1 The Solution: Starting Off

Although this problem is easy to state, a major challenge of solving it is clearly understanding its meaning. What defines a "game" using a fair coin? There seem to be an infinite number of games we could play, with many different kinds of rules! It is always good to start off solving a hard problem by considering **basic examples**, and doing so **slowly**. Often people will brave the tsunami in a shaky boat, erroneously under the impression that they are sailing a sturdy ship. In our case, this ship is our **understanding**. We must first conquer the small waves and strengthen our ship – otherwise, we have no chance of conquering the tsunami!

3.0.2 Basic Games

With this in mind, let's consider some simple values of α, for which we should construct simple games. For example, what if $\alpha = 0$? Can you make a game, using a fair coin, where the probability of me winning is 0? Indeed, consider the following game:

I lose.

This game is not very fun. What if $\alpha = 1$? Can you make a game, using a fair coin, where the probability of me winning is 1? Consider the following game:

I win.

I like this game. We should play it more often. Although seemingly trivial games, they serve as important first steps to constructing a full solution. At least we know how to stay afloat in the kiddie pool. Now, let's make it a bit deeper. What if $\alpha = 0.5$? Can you make a game, using a fair coin, where the probability of me winning is 0.5? Consider the following game:

Flip the coin.

If it flips H, I win. If it flips T, I lose.

Indeed it is clear that for this game, I have a 50% chance of winning. The added twist from before is that we are now using the results of the coin flip, combined with the fact that the coin is a fair coin, to construct our game and obtain our probabilistic guarantee. Pushing further, we can solve the $\alpha = 0.25$ case:

Flip the coin twice.

If it flips HH, I win. Otherwise, I lose.

Of course, here we apply the fact that the coin flips are independent events, and so the probability of flipping HH is $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$. We now see that we can use the results of multiple coin flips to determine the winner. The $\alpha = 0.125$ (i.e. $\alpha = 1/8$) case can be solved similarly:

Flip the coin three times.

If it flips HHH, I win. Otherwise, I lose.

It seems values of α of the form $\alpha = 1/2^k$ can be solved using the natural extension of this strategy:

Flip the coin k times.

If it flips $\underbrace{\text{HHH...HHH}}_{k}$ times, I win. Otherwise, I lose.

What about values of α with denominators of the form 2^k, but with arbitrary numerators? For example, what about $\alpha = 5/8$? We know flipping a fair coin 3 times gives $2^3 = 8$ results of equal probability. Thus, we could just pick 5 of them to be winning for me, and the other 3 to be losing!

Flip the coin three times.

If it flips HHH, HHT, HTH, HTT, or THH, I win. Otherwise, I lose.

Convince yourself that for the above game, I will win with probability exactly $\alpha = 5/8$. This strategy can be generalized to any $\alpha = i/2^k$.

Flip the coin k times.

For some i outcomes, I win. Otherwise, I lose.

Because there are 2^k equally likely outcomes resulting from k coin flips, I will win this game with probability $\alpha = i/2^k$. Our ship can brave an Olympic pool, but there is still a crucial challenge which must be overcome. We must be able to make games for *all* values of α between 0 and 1, but there are still many values of α for which we are stuck. For example, what about $\alpha = 1/3$?

3.0.3 Looping Games

At the moment, our games take the form of flipping a coin some number of times, and then counting some number of the results as "winning" and the rest as "losing". It is hard to see how to move beyond this general structure. What other games can we cook up using a fair coin? Laying on the beach, left with only you, the ocean, and your open mind, you may imagine a game with a completely different feeling.

Flip the coin.

If it flips H, I win. Otherwise, flip again.

This key new rule – "flip again" – will allow us to sail the seas with strength. The game above is a bit silly, because I will always win. I can never lose! Thus such a game solves the $\alpha = 1$ case, which we have already addressed. Despite this redundancy, we can modify it to produce more interesting results:

Flip the coin twice.

If it flips HH, I win. If it flips HT or TH, I lose. Else, flip twice again.

How can we compute the probability of me winning? This can be done directly, by considering the probability I win on the first round, the second round, and so forth, summing all of these probabilities together. Instead, we can be a bit more clever and compute this probability recursively. If α is the probability of me winning, then after the two coin flips I have probability 1/4 of winning immediately, 1/2 of losing immediately, and 1/4 of continuing the game (and later winning with probability α). Thus we can say

$$\alpha = \frac{1}{4} + \frac{1}{4} \cdot \alpha$$

where the left is the probability of me winning by definition, and the right is the recursive computation! Solving for α gives $\alpha = 1/3$. And now, we have broken into new territory. If we want a quick intuitive interpretation of this result, we can see that I win with 1 state (HH), and I lose with 2 states (HT and TH). Further these states are equally likely. Thus the probability of me winning should be $\alpha = 1/3$! This motivates the generalization of our new looping game:

Flip the coin k times.

For some i outcomes, I win. For some j outcomes, I lose. Else, flip k times again.

Our intuitive interpretation gives $\alpha = i/(i+j)$ as my winning probability, and indeed the recursive computation (as well as the direct computation) produces the same result. From here we can see that given any rational $\alpha = a/b$, we can choose i, j, k to make a valid looping game with winning probability α! This is left as an exercise for the reader.

Our ship is strong, and it can brave the high seas. But, it is not yet able to truly conquer the tsunami, and this can be seen by considering what values of α for which we must still construct games. Specifically, we have devised games for rational values of α ... but what about irrational values of α? For example, what about $\alpha = \pi/4$? Looping games cannot work here. As before, we must add a new twist to our game structure, a new element which allows us to break outside of rational α's. This final piece is the hardest to find. To do so, we should begin by deeply considering the limitations of our current strategy. We will eventually motivate our next major change using a deep computational argument.

3.0.4 Abstracting our Current Games

First, let's abstract our current games (basic and looping), to better understand why we're stuck with only producing rational values of α. First, we perform an **initial flipping** section, where we flip the coins k times. Then, we perform a **looping** section, where we keep re-flipping the coins if they produce one of the "looping outcomes." Lastly, we perform a **final decision** section, where we decide whether I win or I lose based on the non-looping outcomes. In our basic games, this looping section was empty – we only flipped the coins k times, and immediately made our final decision! The point though is that all of our current games fit into this general framework.

Initial Flipping

Looping Section

Final Decision

We can elaborate further by including the general instructions for each section.

Initial Flipping: Flip coins.

Looping Section: Repeat: If coins are part of the looping outcomes, flip coins again.

Final Decision: If coins are part of the winning outcomes, I win. Otherwise, I lose.

For example, consider our looping game from before, where we flipped the coin k times, had some i winning outcomes, some j losing outcomes, and some $2^k - i - j$ remaining looping outcomes.

Initial Flipping: Flip the coins k times.

Looping Section: Repeat: If coins are part of the $2^k - i - j$ looping outcomes, flip coins again.

Final Decision: If coins are part of the i winning outcomes, I win.

Otherwise, the coins are part of the j looping outcomes, and I lose.

As mentioned, our more basic, non-looping games fit into this general framework, only with an empty looping section. Any kind of game with this basic structure will have a rational α winning probability of $\alpha = i/(i+j)$, which as previously described can be shown in multiple ways. Through our abstraction, it's clear we must think outside of the box here to conquer irrational α's.

Let's go even deeper. It seems as though there's simply too many α's, and not enough games we can currently make! At the same time, though, there are an infinite number of α's, but also an infinite number of games we could make, by changing the values of i, j, k. How could this be possible? How is it possible for the number of α's to be infinite, and the number of current games to be infinite, and yet it still seems there are not enough games?

Here is where we can utilize mathematical notions of infinity to prove this observation. Such notions are tied to the work of Georg Cantor, Kurt Gödel, and Alan Turing, arguably some of the most important mathematical work of the 19th and 20th century. Crucially, we note there are an **uncountable** number of α's in $[0, 1]$. Are there only a **countable** number of games we can currently design? If so, can we show it?

Indeed, the simple requirement that each game must be described in **finite characters** in English is enough to show the number of games we can currently design is **countable**! The proof is simple and deep. Every game which can be described in a finite number of characters can be encoded as a unique natural number (ex. via conversion of the characters to ASCII). Thus, the number of such games is at most the cardinality of $\mathbb{N}$, which is countable!

3.0.5 Infinite-Length Games

From this observation we can see the only way to conquer irrational α's is to allow games with an **infinite** list of instructions. Such games must still be guaranteed to terminate, though, in order to produce a winner and a loser. This is not necessarily a contradiction, as we will see soon.

Now, we scratch our heads. Suppose we are given an irrational α. For example, suppose we have $\alpha = 0.783\ldots$. How can we somehow encode α, using a single fair coin, into a game, where we allow an infinite number of instructions? Well, the "infinite" part of α comes from its decimal expansion. We want to encode α using a fair coin, which can only flip “heads” or “tails”. What if, instead, we considered the **binary expansion** of α?

$$\alpha = 0.1100100\ldots_2$$

Let’s begin by flipping the coin once, and then think about what makes the most sense. If its Heads, we could say that I win. It should be the case that at least 50% of the time, I should win, so this makes sense as an initial choice. If its Tails, what should happen? Should I lose? No, because then I win with probability 50%. Should I win? No, because then I win with probability 100%. The only option which makes sense is for the game to continue.

Let’s repeat this reasoning for the second coin. If its Heads, we could say that I win. My winning probability is already 50% from the first coin, and this would add another 25% to that total, which is okay because I should win at least 75% of the time. If its Tails, the same reasoning indicates the game should continue.

Now, the third coin. If its Heads, I should lose. This is because a win would add $1/8 = 12.5\%$ to my winning probability, which is too much, since $\alpha < 0.875$. If its Tails, I should continue.

We begin to see the pattern. Reading off the binary expansion of α tells us how to build our game! The ith digit corresponds to the ith coin. If this digit is 0, Heads should mean I lose. If this digit is 1, Heads should mean I win. In all cases, Tails continues me to the next coin.

Given an arbitrary α’s binary expansion

$$\alpha = 0.b_1b_2b_3\ldots_2$$

Our game can be summarized in the following way: repeatedly toss the coin until it comes up Heads. Suppose it comes up Heads on the ith coin. Then if $b_i = 0$, I lose, and if $b_i = 1$, I win. The probability I win on the ith coin is simply $b_i/2^i$ for all i, and so the probability I win on any coin is

$$\sum_{i=1}^{\infty} \frac{b_i}{2^i}$$

but indeed this is **precisely** equal to α! Further, our game **always** ends, because the coin must **always** land on Heads at some point. More precisely, we could say the probability of the game taking N flips is $\left(\frac{1}{2}\right)^N$. As $N \to \infty$, $\left(\frac{1}{2}\right)^N \to 0$. Thus the probability of the game continuing forever is 0, and so it must always end and decide a winner. With this strategy, we have conquered the tsunami, and solved the problem!

Let’s take a look back at how our “infinite instructions” insight was implemented in this problem. Although it may seem our final game structure can be implemented in finite instructions (indeed, I explained it to you in finite instructions!), we need **all** binary digits of the irrational α to fully describe it. Thus indeed the game description requires an infinite amount of information, and this allows us to cross the bridge from countably many games to uncountably many games. Further, note this final game structure works for **all** values of α, not just irrational ones. We can ditch our basic games, and our looping games, and just stick with this game structure!

3.0.6 Conclusion

Although such a problem may seem esoteric (as many do), the broader insights about the initial countability of our games and the uncountability of the α's were noted by Alan Turing in his birth of theoretical computer science. He noted that all "algorithms" must be describable in finite instructions (and thus there are **countably** many), but "problems" do not have this restriction (and thus there are **uncountably** many). Through this observation, he proved that MOST problems we can create are NOT solvable using any algorithm. Kurt Gödel also used this observation to show that most mathematical statements cannot be proven. This is because all "proofs" must be finite in length (and thus there are **countably** many), but "mathematical questions" do not have this restriction (and thus there are **uncountably** many). This was one of the most important mathematical discoveries of the 20th century – arguably *the* most important.

With that, we have solved our coin problem, and conquered the tsunami! We now give many interesting extensions which utilize the power of our final strategy.

4. Extension Solutions

Calm Waves Extension 1

Let $x = 1+\frac{1}{1+\frac{1}{1+\frac{1}{\cdots}}}$ and note $x = 1+\frac{1}{x}$. Solving, we obtain $x = \frac{1\pm\sqrt{5}}{2}$. Of course $x = \frac{1-\sqrt{5}}{2}$ is negative and so that solution is extraneous. Thus, the correct answer is $x = \boxed{\dfrac{1+\sqrt{5}}{2}}$.

Calm Waves Extension 2

(Solution by Ojas Mishra). If one simply wants to see the "cool trick" associated with solving this problem, they should not worry about more rigorous concerns. However, if they want to really understand the formal mathematics, the solution presented skips over some important considerations. Specifically, what exactly does $\sqrt{2+\sqrt{2+\sqrt{\dots}}}$ mean? What does it mean to compute a mathematical expression which is infinite in length? We cannot simply add up the terms, because we will never run out of them!

Rigorously, we define the recursive sequence $a_1 = \sqrt{2}$, $a_{n+1} = \sqrt{2+a_n}$ for all $n \in \mathbb{N}$. If such a sequence converges, we let $L = \lim_{n\to\infty} a_n$. From there, we define $\sqrt{2+\sqrt{2+\sqrt{\dots}}}$ as being this limit L. This makes conceptual sense, given that if one were to try and compute $\sqrt{2+\sqrt{2+\sqrt{\dots}}}$ (as it seems it should be defined), they may do so by repeatedly computing more and more nested square roots. The limit definition of $\sqrt{2+\sqrt{2+\sqrt{\dots}}}$ makes this idea more rigorous.

Thus to show $L = 2$ we must first prove that the sequence converges, and then prove that it converges to 2. To establish convergence, we show $a_n < 2$ for all $n \in \mathbb{N}$, and that a_n is an increasing sequence, both of which can be established by induction. From there we know the sequence converges, and instantiate its limit L. Applying the recurrence, we know $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{2+a_n} \implies L = \sqrt{2+L}$. Now we are allowed to solve the problem in the same way as before, eventually obtaining $L = 2$ as desired.

Rough Waters Extension 1

Connect the centers of the three circles. The common region splits into an equilateral triangle and three "bananas" from before.

$$A = T + 3B = T + 3(S - T) = 3S - 2T$$

$$A = 3S - 2T$$

Solving gives $A = \boxed{\frac{\pi}{2} - \frac{\sqrt{3}}{2}}$, which constitutes $\boxed{\approx 22.4\%}$ of each circle's area.

Rough Waters Extension 2

Connect points A, B, C, D as they were originally defined in the problem. Quadrilateral $ABCD$ is a rhombus with sides of length 1 and a diagonal of length $\sqrt{2}$. Connecting the diagonals of $ABCD$ (which intersect at right angles), we can use simple geometry to see that in fact $ABCD$ is a square. Using the vertical diagonal $\overline{CD}$ to split the overlapping region in half, we can see each half is simply a $90°$ sector minus half of square $ABCD$. Thus the overlapping region has area

$$A = 2\left(\frac{\pi}{4} - \frac{1}{2}\right)$$

$$A = \boxed{\frac{\pi}{2} - 1}$$

which comprises $\boxed{\approx 18.2\%}$ of each circle's area.

Tsunami Extension 1

Let $q = \max(p, 1-p) > \frac{1}{2}$. Call the side which flips with probability q "Tails" and the side which flips with probability $1-q$ "Heads". Running the same game as before, we will flip the coin until it lands Heads, and decide the winner based on the number of flips we've performed in total. The probability of winning given the game takes $i+1$ flips is

$(1-q)q^i c_i$ (i Tail flips with probability q, and then a Head flip with probability $1-q$), where $c_i \in \{0,1\}$ and is up to us to choose. Thus we want to choose $c_0, c_1, \ldots$ such that

$$\alpha = (1-q)\sum_{i=0}^{\infty} c_i q^i$$

where $c_0, c_1, \ldots$ must all be 0 or 1. Give a procedure for doing so. To prove your procedure satisfies the above equation, let

$$\alpha_n = (1-q)\sum_{i=0}^{n} c_i q^i$$

and show $\alpha = \lim_{n\to\infty} \alpha_n$. The coefficients $c_0, c_1, \ldots$ encode the game's instructions in the same way as before. Specifically, flip the coin until it lands Heads. If it lands Heads on the $(i+1)$'st flip, I win if $c_i = 1$, and I lose if $c_i = 0$.

Tsunami Extension 2

Split everyone into two groups, where the probability of a group winning is the sum of the probabilities of its members. Play the standard two-person game with your coin. Repeat this splitting process on the winning group until only one person remains. The fairness of the coin is not relevant for this general multi-person framework.

Tsunami Extension 3

The expected number of flips is

$$E = \sum_{i=0}^{\infty} \frac{i}{2^i}$$

This can be computed by letting

$$f(x) = \sum_{i=0}^{\infty} x^i = \frac{1}{1-x}$$

Differentiating we obtain

$$f'(x) = \sum_{i=0}^{\infty} i x^{i-1} = \frac{1}{(1-x)^2}$$

Multiplying by x gives

$$x f'(x) = \sum_{i=0}^{\infty} i x^i = \frac{x}{(1-x)^2}$$

Plugging in $x = \frac{1}{2}$, we have

$$E = \sum_{i=0}^{\infty} \frac{i}{2^i} = \frac{1/2}{(1-1/2)^2}$$

$$\boxed{E = 2}$$

which makes intuitive sense since given that because the coin has probability 1/2 of terminating on any given flip, we expect the game to terminate in $\frac{1}{1/2} = 2$ flips. For an unbalanced coin with probability $q = \max(p, 1-p) > \frac{1}{2}$ of flipping one side, the expected number of flips is given by

$$E_q = \boxed{\frac{1}{1-q}}$$

which lines up with our intuitive reasoning from before (because we terminate with probability $1-q$ on any given flip). The multi-person game also takes very short time, because it cuts the pool of potential winners in half every iteration. Specifically we expect the multi-person game to take

$$E_q^N = \boxed{\frac{\lceil \log_2 N \rceil}{1-q}}$$

flips. Even for large N, this is remarkably fast. For example, given a multi-person game with a fair coin ($q = 1/2$) involving all people on Earth ($N \approx 8$ billion), we expect to be able to decide a winner in approximately 66 flips. Given a multi-person game with a fair coin ($q = 1/2$) involving all atoms in the Universe ($N \approx 10^{80}$ atoms), we expect to be able to decide a winner in approximately 532 flips. You could assign every atom in this Universe some probability of winning, take a standard coin off the street, and pick a winning atom according to these probabilities in on average 532 flips.