Pittsburgh Interdisciplinary Mathematics Review
Volume 2, December 2024, Pages 49–58
https://doi.org/10.5195/pimr.2024.35

Defining Probabilities Over the Prime Numbers

Oscar Fawcett

(Communicated by Neil MacLachlan and Evan Hyzer)

1. Introduction

This paper will detail my experience solving a problem I stumbled upon while tutoring math and statistics at Cape Fear Community College in Wilmington, North Carolina. Though at first glance the problem seemed simple, I soon realized that the solution would require knowledge about the Riemann zeta function, which is a common fascination for math enthusiasts like myself. To follow the arguments in this paper, all that is necessary is some elementary knowledge about probability and Calculus II. As the first mathematical project I've ever worked on, I learned a lot about math research through this. Hopefully, readers at the beginning of their research career will as well.

2. Finding the Problem

While tutoring probability and statistics at my old community college, a student once asked me: "if something happens infinitely often, why doesn't it have probability 1?". One reasonable response is to point out that there are an infinite number of even numbers, but the probability of an arbitrary integer[1] being even is $\frac{1}{2}$. Though this response satisfied my student, this is a fairly boring probability distribution. All it does is assign a probability of $\frac{1}{2}$ for an integer to be in the set of even numbers, and $\frac{1}{2}$ for an integer to be in the set of odd numbers. So, I was curious to see if I could expand this distribution to be more than a simple coin flip.

We will now go over the expansion process. To begin, we start with the set of positive integers divisible by 2, i.e., $\{2,4,6,8,10,\ldots\}$ and as in the simple distribution, assign the probability of an integer being in this set to be $\frac{1}{2}$. Next, we move to the set of positive integers divisible by 3, i.e., $\{3,6,9,12,15,\ldots\}$. However, there is an overlap between the set of positive integers divisible by 2 and the set of positive integers divisible by 3. So, to avoid over-counting, let's remove the overlap, leaving $\{3,9,15,21,27,\ldots\}$. Since $\frac{1}{3}$ of all numbers are divisible by 3, and we removed half of all those numbers,

[1] The phrase "arbitrary integer" is hand-waving away some measure theory. Indeed, when you specify a probability distribution, you are implicitly defining what an "arbitrary integer" is. In the way that we count, it is intuitive to think about the probability of an arbitrary integer being even as $\frac{1}{2}$, since every other integer is even. However, if we arrange the natural numbers in the following way: $\{1,3,2,5,7,4,\ldots\}$, then it appears that the probability of selecting an even number is $\frac{1}{3}$.

we will assign the probability of an integer being in this set to be $\frac{1}{3}\frac{1}{2} = \frac{1}{6}$. Since all the numbers divisible by 4 are also divisible by 2, we will move straight to the set of numbers divisible by 5: $\{5, 10, 15, 20, 25, \ldots\}$. Again, we will remove the elements that are also in one of the previous two sets, leaving $\{5, 25, 35, 55, 65, \ldots\}$. Since $\frac{1}{5}$ of all numbers are divisible by 5, with $\frac{1}{2}$ of those numbers divisible by 2 and $\frac{1}{3}$ of those numbers divisible by 3, we will assign the probability $\frac{1}{5}\frac{2}{3}\frac{1}{2} = \frac{1}{15}$ for an integer to belong to this set. We continue this process of assigning probabilities for each positive integer.

First, note that this procedure will only assign positive probability to the sets starting with a prime number. Second, observe that the probability of being in the set starting with 2 is exactly the probability that an arbitrary integer is divisible by 2. For the set starting with 3, the probability is exactly the probability of being divisible by 3, but not divisible by 2. For the set starting with 5, it is exactly the probability of being divisible by 5, but not divisible by 2 or 3. Then if we let p_i represent the ith prime number, this probability distribution represents the probability that for some arbitrary integer, p_i is its smallest prime factor. Thus, this probability can be explicitly written via the following function:

$$f(p_i) = \frac{1}{p_i} \prod_{k=1}^{i-1} \left(1 - \frac{1}{p_k}\right)$$

Where $\prod$ is the product symbol for sequences, i.e., $\prod_{k=1}^{n} a_k = a_1 a_2 \ldots a_n$, along with the notation that $\prod_{k=1}^{0} a_k = 1$. Then $f(p_i)$ would be a probability mass function (or PMF). That is, the function that assigns a specific positive probability to the event that p_i is the smallest prime factor for an arbitrary integer. However, is this *actually* a valid probability mass function? By construction, it seems reasonable that it is, but for this to be a valid PMF, $f(p_i)$ has to sum to 1 over all i. That is:

$$\sum_{i=1}^{\infty} \frac{1}{p_i} \prod_{k=1}^{i-1} \left(1 - \frac{1}{p_k}\right) = 1$$

This is not an obvious result: does the above expression actually sum to 1? This paper will detail my pursuit in answering this question definitively.

3. Solving the Problem

My first attempts to solve this problem using some basic Calculus II knowledge failed quite quickly without giving much insight. So, before continuing to try to prove that $\sum_{i=1}^{\infty} f(p_i) = 1$, I began gathering evidence about this series to see if it was even feasible for me to solve this problem. After my initial excitement started to fade, I found myself staring at a very complicated infinite series. I needed to convince myself that this series actually does converge to 1 before I could commit to trying to prove it.

3.1. Empirical Evidence

The first step of learning about this series was to simply calculate it up to some arbitrarily high index. I chose to go up to the 50 millionth prime since my old laptop would not let me go to anything higher. The results of this calculation can be seen in table 1.

Though 50 million seemed like a reasonable number of primes to sum over, I still needed to be careful about efficiently doing the computation, since there is also a significant amount of multiplication happening with each term. To remedy this, I noticed the following relationship:

$$
\begin{aligned}
f(p_i) &= \frac{1}{p_i}\prod_{k=1}^{i-1}\left(1-\frac{1}{p_k}\right)\\
f(p_{i+1}) &= \frac{1}{p_{i+1}}\prod_{k=1}^{i}\left(1-\frac{1}{p_k}\right)\\
&= \frac{1-\frac{1}{p_i}}{p_{i+1}}\prod_{k=1}^{i-1}\left(1-\frac{1}{p_k}\right)\\
&= \frac{p_i-1}{p_{i+1}}f(p_i)
\end{aligned}
$$

This fact was very useful in making the calculations in Table 1 computable in a manageable amount of time.

n	p_n	$f(p_n)$	$\sum_{i=1}^{n} f(p_i)$
1	2	0.5	0.5
2	3	0.16667	0.66667
3	5	0.06667	0.73333
4	7	0.03810	0.77143
5	11	0.02078	0.79221
6	13	0.01598	0.80819
$\vdots$	$\vdots$	$\vdots$	$\vdots$
50,000,000	982,451,653	$2.76 * 10^{-11}$	0.97288

Table 1: Results for the nth partial sum of the function $f(p_i) = \frac{1}{p_i}\prod_{k=1}^{i-1}(1-\frac{1}{p_k})$.

From Table 1, this sum does look like it could be converging to 1. We can also see that the rate of increase slows down quickly, with more than 80% of the mass occurring on the first 6 values. From this, I became more confident in my hypothesis that $\sum_{i=1}^{\infty} f(p_i) = 1$. However, there are plenty of series that appear to converge before eventually diverging to infinity, such as the harmonic series. So, I needed to learn more about the mathematical structure of this series.

3.2. Mathematical Evidence

Let us ignore the prime number aspect of this series for a moment and inspect the properties of $\sum_{i=1}^{\infty} \frac{1}{a_i} \prod_{k=1}^{i-1}(1-\frac{1}{a_k})$ for some sequence $\{a_i\}_{i=1}^{\infty}$. For which sequences will this series converge to 1? If we let $a_i = r$ for all i, the series reduces to the following:

$$\begin{aligned}\sum_{i=1}^{\infty} \frac{1}{r} \prod_{k=1}^{i-1}\left(1-\frac{1}{r}\right) &= \sum_{i=1}^{\infty} \frac{1}{r}\left(1-\frac{1}{r}\right)^{i-1} \\ &= \frac{1}{r}\sum_{i=1}^{\infty}\left(1-\frac{1}{r}\right)^{i-1}\end{aligned}$$

This is a geometric series![2] Thus, if $|1-\frac{1}{r}|<1$, we have the following promising result:

$$\sum_{i=1}^{\infty} \frac{1}{r} \prod_{k=1}^{i-1}\left(1-\frac{1}{r}\right) = \frac{1}{r}\frac{1}{1-(1-\frac{1}{r})} = \frac{r}{r} = 1$$

Continuing this line of investigation, if we let $a_i = i+N$ for some integer $N \geq 1$, then we have:

$$\begin{aligned}\sum_{i=1}^{\infty} \frac{1}{i+N} \prod_{k=1}^{i-1}\left(1-\frac{1}{k+N}\right) &= \sum_{i=1}^{\infty} \frac{1}{i+N} \prod_{k=1}^{i-1}\left(\frac{k+N-1}{k+N}\right) \\ &= \sum_{i=1}^{\infty} \frac{1}{i+N}\frac{N(1+N)(2+N)\dots(i+N-2)}{(1+N)(2+N)\dots(i+N-1)} \\ &= \sum_{i=1}^{\infty} \frac{N}{(i+N)(i+N-1)} = \sum_{i=1}^{\infty} \frac{N}{i+N-1} - \frac{N}{i+N}\end{aligned}$$

This is a telescoping series! Looking at the partial sums of this series, we have:

$$\begin{aligned}\sum_{i=1}^{n} \frac{N}{i+N-1} - \frac{N}{i+N} &= \left(1-\frac{N}{1+N}\right)+\left(\frac{N}{1+N}-\frac{N}{2+N}\right)+\dots+\left(\frac{N}{n+N-1}-\frac{N}{n+N}\right) \\ &= 1-\frac{N}{n+N}\end{aligned}$$

Sending $n \to \infty$, the partial sums converge to 1. Thus, $\sum_{i=1}^{\infty} \frac{1}{i+N} \prod_{k=1}^{i-1}(1-\frac{1}{k+N}) = 1$ for all $N \geq 1$.

From these two examples, I was equipped with the insights necessary to solve this

[2]In fact, if we set $a_i = \frac{1}{p}$ where $p \in (0,1)$, then we get the function $\frac{1}{a_i}\prod_{k=1}^{i-1}(1-\frac{1}{a_k}) = p(1-p)^{i-1}$ which is exactly the probability mass function for the geometric distribution.

problem. Specifically, the last example reducing to a telescoping series was pivotal to the proof that is about to follow.

3.3. The Proof

Before I present the proof, I need to introduce a few facts about the Riemann zeta function. Let $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ with s being some real number. This function plays an important role in one of the biggest open questions in mathematics: the Riemann Hypothesis. To learn more about the Riemann Hypothesis and why it is so important, see [1]. Now, when $s > 1$, it can be shown that $\zeta(s)$ converges to a finite number via the integral test. Conversely, if $s \leq 1$, then $\zeta(s)$ will diverge to positive infinity. Finally, we have its relationship to the prime numbers. Specifically, Euler's product formula for the Riemann zeta function $\zeta(s) = \prod_{k=1}^{\infty}(1 - \frac{1}{p_k^s})^{-1}$. This relationship between the primes and the zeta function can be see through the following argument:

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \dots$$
$$\frac{1}{2^s}\zeta(s) = \frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \frac{1}{8^s} + \dots$$

If we subtract these two equations, it will remove all fractions whose denominator is divisible by 2 from the right side, leaving:

$$\left(1 - \frac{1}{2^s}\right)\zeta(s) = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \dots$$
$$\frac{1}{3^s}\left(1 - \frac{1}{2^s}\right)\zeta(s) = \frac{1}{3^s} + \frac{1}{9^s} + \frac{1}{15^s} + \frac{1}{21^s} + \dots$$

Again, subtract these two expressions to yield:

$$\left(1 - \frac{1}{3^s}\right)\left(1 - \frac{1}{2^s}\right)\zeta(s) = 1 + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{11^s} + \dots$$

Which removes all the remaining fractions on the right side with denominators divisible by 3. Thus, since every integer greater than 1 is divisible by a prime number, if we repeat this process for every prime we will have:

$$\dots\left(1 - \frac{1}{7^s}\right)\left(1 - \frac{1}{5^s}\right)\left(1 - \frac{1}{3^s}\right)\left(1 - \frac{1}{2^s}\right)\zeta(s) = 1$$
$$\zeta(s) = \frac{1}{(1 - \frac{1}{2^s})(1 - \frac{1}{3^s})(1 - \frac{1}{5^s})(1 - \frac{1}{7^s})\dots}$$
$$\zeta(s) = \prod_{k=1}^{\infty}\left(1 - \frac{1}{p_k^s}\right)^{-1} \quad \text{or} \quad \frac{1}{\zeta(s)} = \prod_{k=1}^{\infty}\left(1 - \frac{1}{p_k^s}\right)$$

With these facts established, we can move to a complete proof that $f(p_i)$ is a valid probability mass function.

Theorem 1. *The function* $f(p_i) = \frac{1}{p_i}\prod_{k=1}^{i-1}(1 - \frac{1}{p_k})$ *is a probability mass function on the set of prime numbers.*

Proof. For $f(p_i)$ to be a probability mass function, the following criteria must be met: (1) $f(p_i) \geq 0$ for all p_i, and (2) $\sum_{i=1}^{\infty} f(p_i) = 1$.

The first criterion is automatically satisfied, since all the prime numbers are positive integers greater than 1. For the second criterion, note that we can rewrite $f(p_i)$ as follows:

$$
\begin{aligned}
f(p_i) &= \frac{1}{p_i}\prod_{k=1}^{i-1}\left(1 - \frac{1}{p_k}\right) \\
&= \left(1 - 1 + \frac{1}{p_i}\right)\prod_{k=1}^{i-1}\left(1 - \frac{1}{p_k}\right) \\
&= \left(1 - \left(1 - \frac{1}{p_i}\right)\right)\prod_{k=1}^{i-1}\left(1 - \frac{1}{p_k}\right) \\
&= \prod_{k=1}^{i-1}\left(1 - \frac{1}{p_k}\right) - \left(1 - \frac{1}{p_i}\right)\prod_{k=1}^{i-1}\left(1 - \frac{1}{p_k}\right) \\
&= \prod_{k=1}^{i-1}\left(1 - \frac{1}{p_k}\right) - \prod_{k=1}^{i}\left(1 - \frac{1}{p_k}\right)
\end{aligned}
$$

With this new telescoping representation of $f(p_i)$, observe the partial sums:

$$
\begin{aligned}
\sum_{i=1}^{n} f(p_i) &= \left(\prod_{k=1}^{0}\left(1 - \frac{1}{p_k}\right) - \prod_{k=1}^{1}\left(1 - \frac{1}{p_k}\right)\right) + \cdots + \left(\prod_{k=1}^{n-1}\left(1 - \frac{1}{p_k}\right) - \prod_{k=1}^{n}\left(1 - \frac{1}{p_k}\right)\right) \\
&= \prod_{k=1}^{0}\left(1 - \frac{1}{p_k}\right) - \prod_{k=1}^{n}\left(1 - \frac{1}{p_k}\right) \\
&= 1 - \prod_{k=1}^{n}\left(1 - \frac{1}{p_k}\right)
\end{aligned}
$$

As noted previously, $\prod_{k=1}^{\infty}(1 - \frac{1}{p_k^s}) = \frac{1}{\zeta(s)}$, thus, $\prod_{k=1}^{\infty}(1 - \frac{1}{p_k}) = \frac{1}{\zeta(1)} = 0$. Therefore, since the partial sums converge to 1, $\sum_{i=1}^{\infty} f(p_i) = 1$.

□

And there we have it! Our desired result has thus been proven. In fact, a similar proof technique can be used to prove the following more general theorem:

Theorem 2. *If* $a_i > 1$ *for all* i, *and* $\prod_{k=1}^{\infty}(1 - \frac{1}{a_k}) = c$, *then* $f(a_i) = \frac{1}{1-c}\frac{1}{a_i}\prod_{k=1}^{i-1}(1 - \frac{1}{a_k})$ *is a probability mass function over the set of positive integers.*

Though I will leave it as an exercise to the reader to prove.

4. Further Details

This probability distribution is interesting, but what can we do with it? If we observed data that was generated from this distribution, what kind of analysis can we perform? As statisticians, we often care about *parametric families* and the *moments* of those families. Once these two things are specified, there is a vast literature on how to perform statistical analysis, though much too extensive for me to introduce here. Instead, I will simply give brief motivations to their value and do some calculations for our recently defined probability distribution.

4.1. Parameterizing the Distribution

A parametric family is a probability distribution that is specified by a typically unknown parameter of interest. An example of this would be the distribution for a coin flip, also known the Bernoulli distribution. If you flip a coin, it will come up heads with some probability, call it p, and will come up tails with probability $1-p$. p in this case is the parameter of interest and can be any value between 0 and 1. If you want to check to see if you have a fair coin, you'll need to estimate the value of p, presumably based on data collected from said coin.

Of course, the Bernoulli distribution is a very simple probability distribution, but parameters show up everywhere. Take the normal distribution, which is ubiquitous in statistics. A normal distribution can be parameterized by its mean, often denoted as μ, and its standard deviation, often denoted by σ, both of which are of high importance to practicing scientists.

Now, going back to the probability distribution discussed in this paper, using Theorem 2, we can define a parametric family with the following probability mass function:

$$h(p_i) = \frac{1}{1-\frac{1}{\zeta(s)}} \frac{1}{p_i^s} \prod_{k=1}^{i-1} \left(1 - \frac{1}{p_k^s}\right)$$

For some unknown $s > 0$. Note that when $s = 1$, this function reduces to the probability mass function defined by Theorem 1. Of course, we could have defined a different parametric family, but defining it this way establishes a clearer relationship between this distribution and the Riemann zeta function.

4.2. Moments

To discuss the moments of a probability distribution, we first need to define the expectation of a random variable. For a random variable X that can take on values

$\{x_1, x_2, \ldots,\}$ and has PMF $h(x)$, the the expectation of X is given by:

$$E(X) = \sum_{i=1}^{\infty} h(x_i)x_i$$

Where $E(X)$ is the expectation of X. Note that $E(X)$ need not be finite. From this, we have that the nth moment of X is given by:

$$E(X^n) = \sum_{i=1}^{\infty} h(x_i)x_i^n$$

Moments are very useful at characterizing a probability distribution. the first moment gives us the mean, while the second moment gives us information about the variance, since we have the relationship $V(X) = E(X^2) - E(X)^2$, where $V(X)$ is the variance of X.

Now, let X be a random variable that can take on values $\{p_1, p_2, \ldots\}$ with the PMF $h(p_i)$ defined in the previous section for some $s > 0$. Then the nth moment of X is given by:

$$E(X^n) = \frac{1}{1 - \frac{1}{\zeta(s)}} \sum_{i=1}^{\infty} \frac{1}{p_i^{s-n}} \prod_{k=1}^{i-1} \left(1 - \frac{1}{p_k^s}\right)$$

Calculating the exact value of this series does not seem like an easy task. However, can we determine when the moments are finite?

Before I attempt to answer this question, let us first state a few facts about the prime zeta function, which I will denote as $\psi(s) = \sum_{i=1}^{\infty} \frac{1}{p_i^s}$. Just as for the Riemann zeta function, $\psi(s)$ converges if and only if $s > 1$. The proof of this fact goes beyond the purposes of this paper, but for more details, see here [2]. Now, let's break down this expectation into 3 cases: (1) when $s > 1$, (2) when $s = 1$, and (3) when $s < 1$. Under case (1), since $\frac{1}{\zeta(s)} \leq \prod_{k=1}^{i-1}(1 - \frac{1}{p_k^s}) \leq 1$ for all i, we have:

$$\frac{1}{\zeta(s) - 1}\psi(s-n) \leq \frac{1}{1 - \frac{1}{\zeta(s)}} \sum_{i=1}^{\infty} \frac{1}{p_i^{s-n}} \prod_{k=1}^{i-1} \left(1 - \frac{1}{p_k^s}\right) \leq \frac{1}{1 - \frac{1}{\zeta(s)}}\psi(s-n)$$

Thus, $E(X^n)$ is finite if and only if $s - n > 1$, since $\psi(s-n)$ is finite if and only if $s - n > 1$.

For cases (2) and (3), more involved arguments will be necessary, so for the sake of clarity, we will only discuss some ideas for these cases. Note that as s decreases, $\frac{1}{p_i^s}$ approaches 0 more slowly, while $\prod_{k=1}^{i-1}(1 - \frac{1}{p_k^s})$ approaches 0 more rapidly. Thus, to understand the convergence of $E(X^n)$, we must first understand the growth rate of both of these terms. We already have an understanding of $\frac{1}{p_i^s}$ via the prime zeta function. For $\prod_{k=1}^{i-1}(1 - \frac{1}{p_k^s})$, note the following:

$$\prod_{k=1}^{i-1}\left(1-\frac{1}{p_k^s}\right) = \exp\left(\sum_{k=1}^{i-1}\log\left(1-\frac{1}{p_k^s}\right)\right)$$

To simplify further, we need to bound the term $\log(1-\frac{1}{p_k^s})$. To do this, note the following relationship for logarithms:

$$\frac{x}{1+x} \leq \log(1+x) \leq x$$

When $x > -1$. There are many ways to demonstrate this, but the simplest is to graph these three functions. Once this bound is established, we immediately have:

$$-\frac{1}{p_k^s - 1} \leq \log\left(1-\frac{1}{p_k^s}\right) \leq -\frac{1}{p_k^s}$$

Applying this to our product term then yields:

$$\exp\left(-\sum_{k=1}^{i-1}\frac{1}{p_i^s - 1}\right) \leq \prod_{k=1}^{i-1}\left(1-\frac{1}{p_k^s}\right) \leq \exp\left(-\sum_{k=1}^{i-1}\frac{1}{p_i^s}\right)$$

Hence, to understand when $\sum_{i=1}^{\infty}\prod_{k=1}^{i-1}(1-\frac{1}{p_k^s})$ converges, it is enough to understand:

$$\sum_{i=1}^{\infty}\exp\left(-\sum_{k=1}^{i-1}\frac{1}{p_k^s}\right)$$

When $s > 1$, the above series will definitely diverge, since $\exp(-\prod_{k=1}^{\infty}\frac{1}{p_k^s}) = \exp(-\psi(s))$ which is just some positive number. When $s = 1$, note that $\sum_{i=1}^{n}\frac{1}{p_i} \leq \sum_{i=1}^{p_n}\frac{1}{i} = H_{p_n}$ where H_n is known as the nth harmonic number. There is much literature on harmonic numbers, including the following bounds:

$$\log(n) \leq H_n \leq \log(n) + 1$$

For more details on harmonic numbers, see [3]. Putting these two bounds together, we have:

$$\sum_{i=1}^{\infty}\exp\left(-\sum_{k=1}^{i-1}\frac{1}{p_k}\right) \geq \sum_{i=1}^{\infty}\exp(-H_{p_{i-1}}) \geq \sum_{i=1}^{\infty}\exp(-log(p_{i-1})-1)$$
$$\geq \sum_{i=1}^{\infty}\exp(-1)\frac{1}{p_{i-1}} = \infty$$

The above argument also provides a proof that $E(X) = \infty$ when $s = 1$. When $s < 1$, my hunch is that $\sum_{i=1}^{\infty}\exp(-\sum_{k=1}^{i-1}\frac{1}{p_k^s})$ will converge, but unfortunately I don't have an argument for this at the time of writing.

Now that we have some idea as to when the moments of X exist, we can get approximations for them using the same approach as in section 3.1. That is, using a computer to calculate the partial sum up to some arbitrarily high index.

5. Conclusion

In this paper, we proved the validity of a novel probability distribution over the prime numbers. In doing so, we also provided a way to define probability distributions over more general sequences $\{a_i\}_{i=1}^{\infty}$. In chronicling my process to arrive at these results, I hope to have provided useful insights to those at the start of their mathematical research careers.

References

[1] Kontorovic, Alex. "The Riemann Hypothesis, Explained" YouTube, uploaded by Quanta Magazine, https://www.youtube.com/watch?v=zlm1aajH6gY

[2] Weisstein, Eric W. "Prime Zeta Function." From MathWorld–A Wolfram Web Resource. https://mathworld.wolfram.com/PrimeZetaFunction.html

[3] Sondow, Jonathan and Weisstein, Eric W. "Harmonic Number." From MathWorld–A Wolfram Web Resource. https://mathworld.wolfram.com/HarmonicNumber.html

Department of Statistics, University of Pittsburgh, Pittsburgh, PA 15260
E-mail address: osf8@pitt.edu