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The Category of Graphs

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Abstract

The category of graphs and the mappings between them is considered. The monomorphisms and epimorphisms are characterized. Reflective and coreflective subcategories are identified and terminal, initial, projective, and injective objects are characterized. Parallels with the category of topological spaces are discussed.

1. Introduction.

A *graph* G is an ordered pair $G = (V,E)$ where V is a non-empty set of elements called *vertices* and E is a set, possibly empty, of pairs of distinct elements of V, called *edges*. An edge $\{v,w\}$, usually written simply vw, is said to be *incident* on the vertices v and w. We assume that there is at most one edge incident on any pair of vertices.

A graph H is a *subgraph* of a graph G if every vertex and every edge of H is also a vertex or edge of G.

A *graph morphism* f from G to G' is a pair of morphisms in the category $\mathcal{S}$ of sets $f_V : V \to V'$ and $f_E : E \to E'$ such that f_E preserves incidence, i.e., $f_E(\{u,v\})$ is the edge $\{f_V(u), f_V(v)\}$ in G'. A morphism i is said to be an *isomorphism* if i_V and i_E are both one-to-one and onto.

The category $\mathcal{G}$ of graphs includes graphs as its objects and graph morphisms as its morphisms. $\mathcal{G}$ is a concrete category since it has a forgetful functor to the category $\mathcal{S}$ of sets and mappings.

One goal of category theory is to draw parallels between areas of mathematics. Here we examine the parallels between reflective and coreflective subcategories of $\mathcal{G}$ and such categories in the category of topological spaces.

The details from topology can be found in [3], and a shorter account is in [4].

2. Morphisms in $\mathcal{G}$.

In any category a *monomorphism* is a morphism $m : B \to C$ such that if f and g are morphisms from A to B such that $m \circ f = m \circ g$ then $f = g$.

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The dual is an *epimorphism*. A morphism $e : A \to B$ is an *epimorphism* if for morphisms f and g from B to C the equality $f \circ e = g \circ e$ implies that $f = g$.

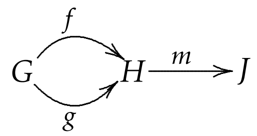
In diagrams a monomorphism is indicated by an arrow with a tail, while an epimorphism is indicated by an arrow with two heads as below:

$$G \overset{m}{\rightarrowtail} H \overset{e}{\twoheadrightarrow} J$$

Proposition 2.1. *The monomorphisms in $\mathcal{G}$ are the morphisms where f_V is one-to-one.*

Proof. Let $m_V : V(H) \to V(J)$ be one-to-one. We first show that m_E is also one-to-one. If uv and xw are distinct edges, then we can assume that $u \neq x$. Then $f_V(u) \neq f_V(x)$ so that $\{f_V(u), f_V(v)\} \neq \{f_V(x), f_V(w)\}$.

Now assume that $m \circ f = m \circ g$ in the diagram below:

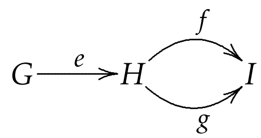


Then for a vertex v of G and vw an edge of G, $m_V(f_v(v)) = m_V(g_v(v))$ and $m_E(f_E(uw)) = m_E(g_E(uw))$. Since m_V and m_E are one-to-one, $f_V(v) = g_V(v)$ and $f_E(uw) = g_E(uw)$ so that $f = g$ and m is a monomorphism.

Conversely, let m be a monomorphism and let v_1 and v_2 be distinct vertices in H. Let G be the trivial graph K_1 with the single vertex u, and define f and g from K_1 to H by $f_V(u) = v_1$ and $g_V(u) = v_2$. Because m is a monomorphism, $m \circ f \neq m \circ g$. Hence, $m_V(v_1) \neq m_V(v_2)$ so that m_V is one-to-one. The fact that m_E is one-to-one follows as shown above. □

Proposition 2.2. *The epimorphisms in $\mathcal{G}$ are the morphisms where f_V is onto.*

Proof. Let $e : G \to H$ have e_V onto and let $f \circ e = g \circ e$ in the diagram below:



Let v be a vertex in H, and choose u in $e_V^{-1}(v)$. Then $f_V \circ e_V(u) = g_V \circ e_V(u)$ so that $f_V(v) = g_V(v)$. The equality of f_E and g_E follows from the requirement that f_E and g_E preserve incidence. Hence, e is an epimorphism.

For the converse we prove the contrapositive. Let $e : G \to H$ and assume that there is a vertex v_0 in H which is not in the image of G.

Consider first the case where H has only two vertices v_0 and v_1. Define H' to be H with an additional vertex v_2. If v_0 has degree 0, add the edge v_1v_2, if v_0 has degree 1 in H, let v_2 have degree 0 in H'. Then define $f : H \to H'$ by $f_V(v_1) = v_1$ and $f_V(v_0) = v_2$ and define $g : H \to H'$ by $g_V(v_1) = v_1$ and $g_V(v_0) = v_0$. Then the degrees of $f_V(v_0)$ and $g_V(v_0)$ are different, yet $g \circ e = f \circ e$, so e cannot be an epimorphism.

Now assume that H has three or more vertices.

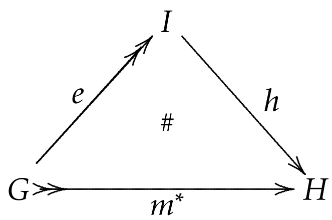
Consider the case where a missed vertex v_0 has degree 0. Let H' be H with v_0 replaced by an isolated v_1 and a new vertex v_2 with an edge v_2w where w is part of the image of G. Define $f : H \to H'$ with $f_V(v_0) = v_1$ and the identity otherwise; and $g : H \to H'$ the same except $g_V(v_0) = v_2$. Then $g \neq f$ but $g \circ e = f \circ e$, so e cannot be an epimorphism.

Now suppose that the missed vertex v_0 has degree 1, and let v_0w be its edge in H. Define $f : H \to H$ to be the identity and $g : H \to H$ be the identity on H other than on v_0 and v_0w so that $f_V(v_0) = w$ and the edge disappears. Then $g \circ e = f \circ e$, but $f \neq g$, so e cannot be an epimorphism.

Now consider the case where $\deg(v_0) \geq 2$. Let u and w be vertices adjacent to v_0. Let H' be H with edges added joining the vertices adjacent to v_0 if such edges do not exist in H. Define $f : H \to H'$ to be the identity on H except that $f(v_0) = u$, the edge v_0u is lost, and the other edges incident on v_0 go to the correct, perhaps added, edges of H'. Define $g : H \to H'$ similarly except $g(v_0) = w$ and the edge v_0w is lost. Then $g \circ e = f \circ e$ but $f \neq g$, so e cannot be an epimorphism.

Hence, the epimorphisms in $\mathcal{G}$ are the morphisms f for which f_V is onto. □

A monomorphism m^* is an *extremal monomorphism* if when m^* is factored as below

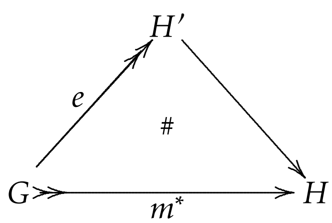


and e is an epimorphism, then e is an isomorphism. An extremal monomorphism is indicated in a diagram by an arrow with a double tail.

A subgraph H of a graph G is a called an *induced subgraph* if for u and v vertices of H and uv an edge in G, then uv is also an edge in H. Thus an induced subgraph of G can be defined by choosing the vertices and then requiring that all edges of G which are incident on a pair of chosen vertices also belong to the subgraph.

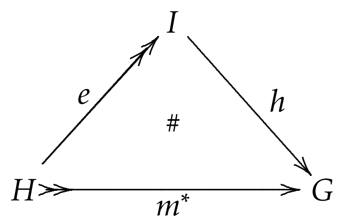
Proposition 2.3. *The extremal monomorphisms in $\mathcal{G}$ are the embeddings of subgraphs induced by a set of vertices.*

Proof. We first show that an extremal monomorphism is the embedding of a subgraph induced by a set of vertices. Let $m^* : G \to H$ be an extremal monomorphism. Then we can factor m^* through the subgraph H' induced by $m^*_V(G)$ in H as diagramed below:



e is an epimorphism since e_V is onto. Hence, e is an isomorphism, and m^* is the embedding of an induced subgraph.

Now let m^* be the embedding into G of the subgraph H induced by the set S of vertices of G, and let $m^* = h \circ e$ where e is an epimorphism:



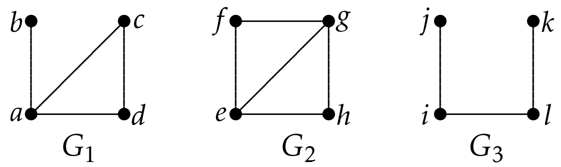
Let v_0, v_1 be distinct vertices in H, and suppose that $e(v_0) = e(v_1)$. Then $h_V \circ e_V(v_0) = h_V \circ e_V(v_1)$ which is impossible because $m^* = h \circ e$ and m^* is an embedding, hence, one-to-one. Thus, e_V is one-to-one, and e_E is also one-to-one since it preserves incidence.

Now suppose that uv is an edge in I not in the image of e_E. Then $\{h_E(u), h_E(v)\}$ is an edge in G incident on vertices in the image of H. But H is an induced subgraph, so the corresponding edge is present in H, and hence in the image of e in I. Thus, e_V is also onto, and e is an isomorphism. Hence, m^* is an extremal monomorphism. □

If $m^* : G \to H$ is an extremal monomorphism, then G is said to be an *extremal subobject* of H. So in $\mathcal{G}$, the extremal subobjects are subgraphs induced by a set of vertices.

Example 2.4. A monomorphism that is not an extremal monomorphism.

Consider the three graphs below:



Define $m : G_3 \to G_2$ by $j \mapsto f, i \mapsto e, l \mapsto h, k \mapsto g$, define $e : G_3 \to G_1$ by $j \mapsto b$, $i \mapsto a, l \mapsto d, k \mapsto c$, and define $f : G_1 \to G_2$ by $b \mapsto f, a \mapsto e, d \mapsto h, c \mapsto g$. The definitions of m, e, and f on edges are determined by preservation of incidence.

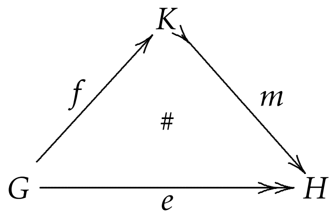
Then $m = f \circ e$, m is a monomorphism, and e is an epimorphism since e_V is onto, but m is not an isomorphism. Hence, m is not an extremal monomorphism. ■

A morphism e is an *extremal epimorphism* if it can be factored as $e = m \circ f$ where m is a monomorphism, then m is an isomorphism. An extremal epimorphism is denoted in a diagram by an arrow with a triple head.

Proposition 2.5. *The extremal epimorphisms in $\mathcal{G}$ are the morphisms e where both e_V and e_E are onto.*

Proof. Let $e : G \to H$ have both e_V and e_E onto, and let the following diagram commute with

m a monomorphism:

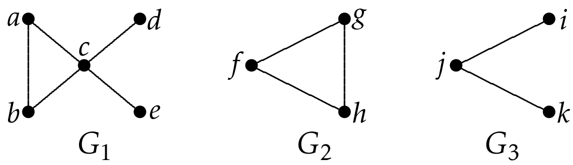


Because e_V and e_E are both onto and the diagram commutes, m_V and m_E must both be onto. m_V is also one-to-one since m is a monomorphism. It follows that m_E is one-to-one since it preserves incidence, m_V is one-to-one, and there is at most one edge incident on a pair of vertices. Hence, m is an isomorphism, and therefore e is an extremal epimorphism.

Now suppose that $e: G \twoheadrightarrow H$ is an extremal epimorphism. Let K be the subgraph of H induced by the vertices in the image $\{e_V(v) : v \text{ a vertex of } G\}$ and let m be the embedding of K into H. Define $f : G \to K$ by $f_V(v) = u$ if $e_V(v) = u$. Then by construction $e_V = m_V \circ f_V$. It follows that $e_E = m_E \circ f_E$ by preservation of incidence. Then m is a monomorphism, so by the assumption that e is an extremal epimorphism, m is an isomorphism, and m_V and m_E are onto, and thus so also are e_V and e_E. □

Example 2.6. An epimorphism that is not an extremal epimorphism.

Consider the three graphs below:



Define $e : G_1 \to G_2$ by $a \mapsto f, b \mapsto f, c \mapsto f, d \mapsto g, e \mapsto h$, define $m : G_3 \to G_2$ by $i \mapsto g, j \mapsto f, k \mapsto h$, and define $g : G_1 \to G_3$ by $a \mapsto j, b \mapsto j, c \mapsto j, d \mapsto i, e \mapsto k$.

Then $e = m \circ g$ and m is a monomorphism but not an isomorphism. e_V is onto, so e is an epimorphism, but e_E is not onto so e is not an extremal epimorphism. ■

A graph is said to be *complete* if its set of edges includes all possible pairs of vertices. A sequence of vertices $v_0, v_1, v_2, \ldots, v_n$ together with edges $v_0v_1, v_1v_2, \ldots, v_{n-1}v_n$ incident on consecutive pairs of vertices in the sequence is called a *path* connecting v_0 and v_n. A graph G is *connected* if there is a path in G connecting any pair of vertices in G.

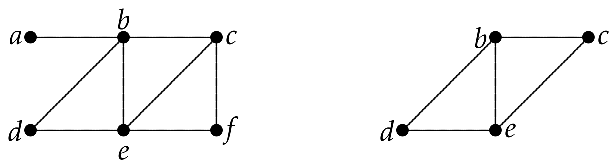
Proposition 2.7. *If G is connected or complete and $e : G \to H$ is an epimorphism, then H is connected or complete, respectively.*

Proof. Let G be connected and h and l be vertices in H. Then for u in $e_V^{-1}(h)$ and v in $e_V^{-1}(l)$ there is a path in G connecting u and v. The images under e of the vertices and edges on that path form a path connecting h and l, and H is therefore connected.

Now let G be complete and h and l be distinct vertices in H. Since G is complete there is an edge incident on any pair of vertices with one drawn from $e_V^{-1}(h)$ and one from $e_V^{-1}(l)$, and the image of that edge is the edge hl in H, so H is complete. □

A *contraction* is a morphism which identifies the vertices incident on an edge, or a set of edges.

Example 2.8. Consider the graphs below:



The morphism described by

$$a \mapsto b,\, b \mapsto b,\, c \mapsto c,\, d \mapsto d,\, e \mapsto e,\, f \mapsto e$$

has contracted the vertices on the edge ab and the edge ef. ■

A contraction is clearly an extremal epimorphism, and the fact that the converse is true provides a characterization of extremal epimorphisms.

Proposition 2.9. The extremal epimorphisms in the category of graphs are the contractions on a set of edges.

Proof. Because for a contraction f both f_E and f_V are onto a contraction is an extremal epimorphism.

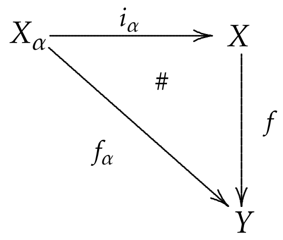
Now let $e : G \to H$ be an extremal epimorphism. Let $\{h_i\}$ be the set of vertices in H such that $e_V^{\leftarrow}(h_i)$ is not a singleton. Then e_V has identified the vertices incident on any edges in G which join vertices in $e_V^{\leftarrow}(h_i)$. Any edges of G which join a vertex v not in $e_V^{\leftarrow}(h_i)$ with one in $e_V^{\leftarrow}(h_i)$ are assigned by e to the edge from the image of v to h_i.

Pairs of vertices v of H for which $e_V^{\leftarrow}(v)$ is a singleton and which have an edge incident on them are the images of a distinct pair of vertices of G and the edge between them. Hence both e_v and e_E are one-to-one except for the images associated with the contraction of a set of edges. □

3. Constructions in the category $\mathcal{G}$.

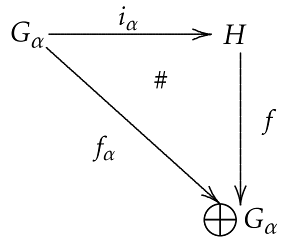
The *sum* of a family of graphs with disjoint sets of vertices is the graph with vertex and edge sets the union of the vertex and edge sets of the members of the family.

The *coproduct* of a set $\{X_\alpha\}$ of objects in a category $\mathcal{C}$ is an object X together with a family of morphisms $i_\alpha : X_\alpha \to X$ such that for any other object Y and family of morphisms $f_\alpha : X_\alpha \to Y$ there exists a unique morphism $f : X \to Y$ such that the following diagram commutes for each α:



Proposition 3.1. *The coproduct of a family of graphs in $\mathcal{G}$ is their sum.*

Proof. Let $\{G_\alpha\}$ be a family of graphs, let H be the coproduct of the family and let $\bigoplus G_\alpha$ be the sum of the graphs. Then there are morphisms $i_\alpha : G_\alpha \to H$ and the injections $f_\alpha : G_\alpha \to \bigoplus G_\alpha$. By the defining property of the coproduct there exists a unique morphism $f : H \to \bigoplus G_\alpha$ such that the following diagram commutes:



We show that f is an isomorphism.

We first show that f_V is onto. Let v be a vertex of $\bigoplus G_\alpha$. Then $v = f_{\alpha_V}(v_\alpha)$ for some α, and since the diagram commutes, $v = f_{\alpha_V}(v_\alpha) = f_V(i_{\alpha_V}(v_\alpha))$, and thus f_V is onto.

To see that f_V is one-to-one, let v and u be distinct vertices in H. There are two possibilities. First, if $f_V(v)$ and $f_V(u)$ are in distinct graphs G_α and G_β, then clearly $f_V(v) \neq f_V(u)$. Otherwise, $f_V(v)$ and $f_V(u)$ both belong to G_α for some α. Then $u = i_{\alpha_V}(u_\alpha)$ and $v = i_{\alpha_V}(v_\alpha)$ for some u_α and v_α in G_α. Then

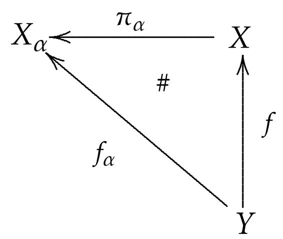
$$f_V(v) = f_V(i_{\alpha_V}(v_\alpha) = f_{\alpha_V}(v_\alpha) \neq f_{\alpha_V}(u_\alpha) = f_V(i_{\alpha_V}(u_\alpha) = f_V(u)$$

since f_α is one-to-one, and hence f_V is one-to-one.

Now consider f_E. Any edge uv in $\bigoplus G_\alpha$ belongs to some G_α, and there is a corresponding edge $u_\alpha v_\alpha$ in G_α such that $f_{\alpha_E}(u_\alpha v_\alpha) = uv$. Then $f_E(i_{\alpha_E}(u_\alpha v_\alpha)) = uv$ so f_E is onto. To see that f_E is one-to-one, note that the vertices associated with distinct edges in H are mapped to distinct vertices in $\bigoplus G_\alpha$, hence f_E is one-to-one.

Thus, f is an isomorphism in $\mathcal{G}$, and hence the coproduct of a family of graphs in $\mathcal{G}$ is their sum. □

The *product* of a set $\{X_\alpha\}$ of objects in a category $\mathcal{C}$ is an object X together with a family of morphisms $\pi_\alpha : X \to X_\alpha$, called *projections*, such that for any other object Y and family of morphisms $f_\alpha : Y \to X_\alpha$ there exists a unique morphism $f : Y \to X$ such that the following diagram commutes for each α:



The *product* $\times G_\alpha$ of a family of graphs is the graph with vertex set the product $\times V(G_\alpha)$ of the vertex sets of the members of the family in the category of sets. Then a vertex $(v_1, v_2, \ldots, v_\beta)$ is adjacent to a vertex $(u_1, u_2, \ldots, u_\beta)$ if and only if v_i is adjacent to u_i for all i.

Now define the *projection morphisms* $\pi_\alpha : \times G_\alpha \to G_\alpha$ by $\pi_{\alpha_V}(v_1, v_2, \ldots, v_\beta) = v_\alpha$ and $\pi_{\alpha_E}(\{(v_1, v_2, \ldots, v_\beta), (u_1, u_2, \ldots, u_\alpha)\}) = \{v_\alpha, u_\alpha\}$.

Proposition 3.2. The product of graphs defined as above is the product in the category $\mathcal{G}$.

Proof. Consider graphs $\{G_\alpha\}$ and a graph H with morphisms $f_\alpha : H \to G_\alpha$. Then we need to define a morphism $f : H \to \times G_\alpha$ so that $\pi_\alpha \circ f = f_\alpha$. Define f by

$$f_V(v) = (f_{1_V}(v), f_{2_V}(v), \ldots, f_{\beta_V}(v))$$

and

$$f_E(\{u, v\}) = \{(f_{1_V}(u), f_{2_V}(u), \ldots, f_{\beta_V}(u)), (f_{1_V}(v), f_{2_V}(v), \ldots, f_{\beta_V}(v))\}.$$

Then

$$\pi_{\alpha_V}(f_V(u)) = \pi_{\alpha_V}(f_{1_V}(v), f_{2_V}(v), \ldots, f_{\beta_V}(v)) = f_{\alpha_V}(v)$$

and

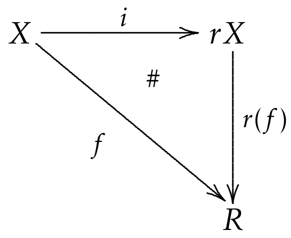
$$\begin{aligned} \pi_{\alpha_E}(f_E(\{u, v\})) &= \pi_{\alpha_E}\Big((f_{1_V}(u), f_{2_V}(u), \ldots, f_{\beta_V}(u)), (f_{1_V}(v), f_{2_V}(v), \ldots, f_{\beta_V}(v))\Big) \\ &= \Big\{f_{\alpha_V}(u), f_{\alpha_V}(v)\Big\} \\ &= f_{\alpha_E}(\{u, v\}) \end{aligned}$$

so that $\pi \circ f = f_\alpha$. □

Note that with this definition of product – one of several possible in $\mathcal{G}$ – the product of complete graphs is complete.

4. Reflective subcategories of $\mathcal{G}$.

A subcategory $\mathcal{R}$ of a category $\mathcal{C}$ is said to be *reflective* if every object X of $\mathcal{C}$ has a *reflection* rX in $\mathcal{R}$ and a morphism $i : X \to rX$ such that if R is an object of $\mathcal{R}$ and there is a morphism $f : X \to R$, then there is a morphism $r(f)$ such that the diagram below commutes:



The reflective subcategory is called *epi-reflective* if the morphism $i : X \to rX$ is an epimorphism.

In this section we consider two subcategories of $\mathcal{G}$ and the extent to which they parallel the reflective subcategory of compact Hausdorff spaces in the category of completely regular spaces.

The set of complete graphs together with the graph morphisms between them constitutes a subcategory $\mathcal{K}$ of $\mathcal{G}$. The complete graph with n vertices is denoted by K_n. It can easily be shown that K_n has $C(n,2)$ edges.

The *completion* kG of a graph G is obtained by adding edges to G until it is complete. Since any two complete graphs with the same number of vertices are easily seen to be isomorphic, we can consider the completion of a graph G with n vertices to be K_n.

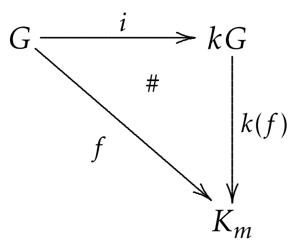
Theorem 4.1. *$\mathcal{K}$ is an epi−reflective subcategory of $\mathcal{G}$.*

Proof. Consider the embedding i of G into kG. i is an epimorphism since i_V is onto.

If $f : G \to K_m$ is any morphism from G to a complete graph, define $k(f) : kG \to K_m$ by

$$\begin{aligned} k(f)_V(v) &= f_V(v) \\ k(f)_E(\{u,v\}) &= \{f_V(u), f_V(v)\}. \end{aligned}$$

Note that the edge $\{f_V(u), f_V(v)\}$ exists in K_m even if $\{u,v\}$ is not an edge in G since K_m is complete. Then $k(f)$ agrees with f on the image of G, and extends f to the added edges in kG so that $k(f)$ is well-defined and the diagram below commutes:



□

In topological spaces the category of compact Hausdorff spaces is a reflective subcategory of the category of completely regular spaces and the morphism between a completely regular space and its reflection is an epimorphism. Such categories are characterized as being closed under products and extremal monomorphisms.

Note that i in the theorem above is an epimorphism, and that we observed previously that the product of complete graphs is complete. Since it is easy to show that an induced subgraph of a complete graph is complete, we have verified the next proposition to establish the parallel with the category of compact Hausdorff spaces.

Proposition 4.2. *The category of complete graphs is closed under products and extremal subobjects.*

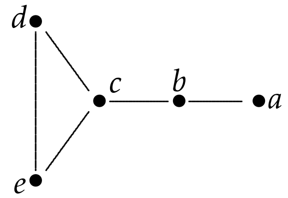
If we restrict the morphisms we allow, we can identify a second reflective subcategory. The *degree* of a vertex in a graph is the number of edges incident on the vertex. A graph is said to be *r-regular* if all vertices in the graph have degree r.

From Theorem 2.7 of [1] we have the following result regarding regular graphs:

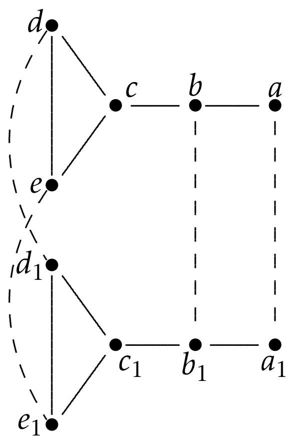
Proposition 4.3. *A graph G with a maximum vertex degree r can be embedded as an induced subgraph of an r-regular graph rG.*

The process of embedding often produces a graph with many additional vertices as the next example demonstrates.

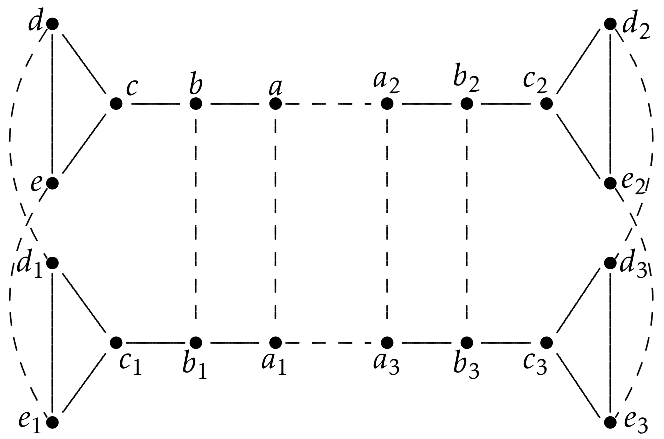
Example 4.4. Consider the graph G below with maximum degree 3 and minimum degree 1:



The 3-regular graph generated by the proposition above is obtained by recursively connecting the vertices of G with degree less than 3 with the corresponding vertices in another copy of G. The first graph obtained in this process has its minimum degree raised to 2 and is shown below. The added vertices are subscripted by 1's and the edges added between copies of G are dashed.



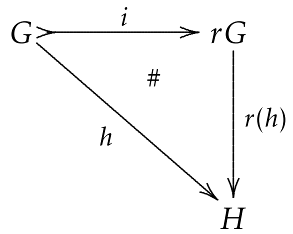
Repeating the graph duplicating process to raise the degrees of a and a_1 yields the 3-regular graph below with G as one of four isomorphic induced subgraphs.



■

A key property of rG is that morphisms from G can be extended to rG.

Theorem 4.5. *If G is a graph with maximum degree r, rG is the r-regular graph as above, and $h : G \to H$ is a morphism, then h can be extended to rG so that the diagram below commutes*:



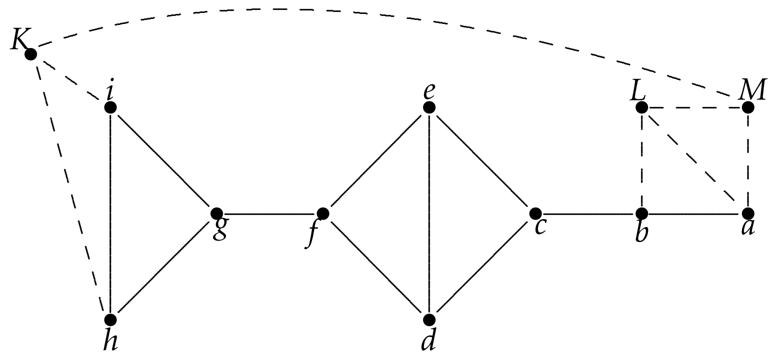
Proof. For each vertex v in G, let the corresponding vertices in the copies of G be denoted by v_i. Define $r(h)_V(v_i) = h_V(v)$. Then $r(h)_V$ is obviously an extension of h_V to $V(rG)$. Note that since a vertex v and all of its copies are sent to the same vertex in H, the edges joining v and its copies are collapsed into the vertex $h_V(v)$. Then $r(h)_E$ is defined by sending the edges u_iv_i to the image of uv under h.□

Theorem 4.6. *The $r-$regular graphs are a reflective subcategory of the category of graphs with maximum degree r.*

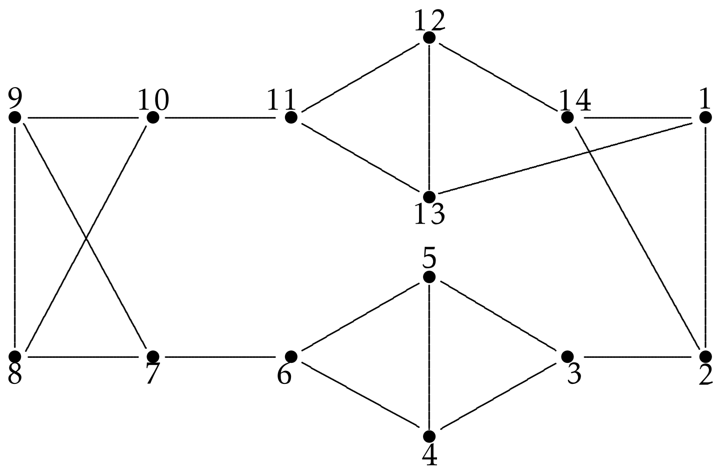
Proof. Let H be an $r-$regular graph. For a graph G with maximum degree r the previous theorem shows that a morphism $h : G \to H$ will extend to the $r-$regular graph rG containing G. Hence the category of $r-$regular graphs is a reflective subcategory of the category of graphs with maximum degree r. □

Since the process of embedding a graph in a r-regular graph as demonstrated above often generates a much larger graph, it is natural to consider smaller r-regular graphs containing a given graph. The next example shows that this might not be a successful direction if the extension of morphisms is an objective.

Example 4.7. The graph G' below is a 3-regular graph. Vertices a through i and the solid edges joining them form a subgraph G that is not 3-regular. Following [2] the vertices K, L and M and the dashed edges have been added to G to create G', the smallest 3-regular graph containing G as an induced subgraph.



The graph H below is a 3-regular graph with vertices 1 through 14.

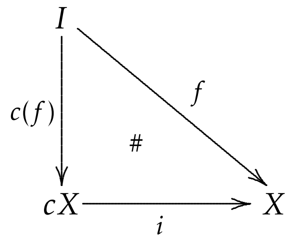


Define a morphism f from G to the graph H by $a \mapsto 1, b \mapsto 2, c \mapsto 3, d \mapsto 4, e \mapsto 5,$ $f \mapsto 6, g \mapsto 7, h \mapsto 8,$ and $i \mapsto 9$.

We show that f cannot be extended to G'. To preserve incidence, K would need to be mapped to one of the vertices 7, 8, 9 or 10, and M would need to be mapped to one of 1, 2 or 14. But then the edge $\{K, M\}$ would need to be mapped to an edge $\{x, y\}$ where $x \in \{7, 8, 9, 10\}$ and $y \in \{1, 2, 14\}$. But the graph H includes no such edge. ■

5. A coreflective subcategory of $\mathcal{G}$.

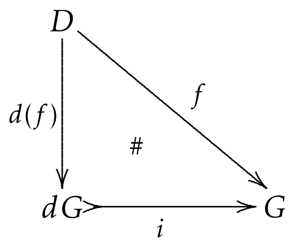
A subcategory $\mathcal{I}$ of a category $\mathcal{C}$ is a *coreflective subcategory* if for any object X in $\mathcal{C}$ there is an object cX in $\mathcal{I}$ and a morphism $i : cX \to X$ such that if I is an object in $\mathcal{I}$ and there is a morphism $f : I \to X$, then there is a morphism $c(f)$ such that the diagram below commutes:



If $i : cX \to X$ is a monomorphism, then $\mathcal{I}$ is a *mono-coreflective subcategory*.

Theorem 5.1. *The category of graphs without edges is a mono-coreflective subcategory of* $\mathcal{G}$.

Proof. For a graph G, let dG be the graph obtained from G by deleting all its edges. Define a morphism $i : dG \to G$ by letting i_V be the identity on $V(G)$. Then i is a monomorphism. For f any morphism from a graph D without edges to G, we need to obtain a morphism $d(f)$ from D to dG. For a vertex v in G, consider all u in the pre-image of v in D, and define $d(f)_V(u) = v$ in dG. It is easy to see that this definition includes all vertices of D, and makes the diagram below commute so that dG is the coreflection of G.



□

In the case of the mono-coreflective category of graphs without edges there is a direct analogy with the category of topological spaces. In that category the coreflective subcategories are closed under topological sums and quotient maps – which are the extremal epimorphisms in the category of topological spaces.

The subcategory of graphs without edges exactly parallels the case of the coreflective topological subcategories

Proposition 5.2. *The category of graphs without edges is closed under sums and images of extremal epimorphisms.*

Proof: The sum of a family of graphs without edges clearly has no edges.

Now let G be graph without edges and let $e: G \twoheadrightarrow H$ be an extremal epimorphism. Then every edge of H is the image of an edge of G. Since G has no edges, neither does H. □

6. Initial and terminal objects in $\mathcal{G}$.

An object T in a category $\mathcal{C}$ is a *terminal object* if for any other object A there is a unique morphism from A to T. An object I in a category $\mathcal{C}$ is an *initial object* if for any other object B there is a unique morphism from I to B. In $\mathcal{G}$ the empty graph is the initial object, and the trivial graph consisting of a single vertex is the terminal object.

The empty graph and trivial graphs are not very interesting. But if we alter the definitions to replace the unique morphism by non-trivial morphisms we can uncover some interesting characteristics of certain types of graphs.

We define an object T in a category $\mathcal{C}$ to be a *full terminal object* if for any other object A the forgetful functor F from $Morph_{\mathcal{C}}(A,T)$ to $Morph_{\mathcal{S}}(F(A),F(T))$ is onto. Similarly, an object I in a category $\mathcal{C}$ is an *full initial object* if for any other object B the forgetful functor F from $Morph_{\mathcal{C}}(I,B)$ to $Morph_{\mathcal{S}}(F(I),F(B))$ is onto.

Proposition 6.1. *The full initial objects in the category $\mathcal{G}$ are the graphs without edges.*

Proof: Let G be a full initial object in $\mathcal{G}$ and let H be the graph with two vertices h_1 and h_2 and no edges. Then for every pair of vertices v and u in G there is a function $f : F(G) \to F(H)$ such that $f(v) = h_1$ and $f(u) = h_2$. The pre-image f' of f under F is a morphism $f' : G \to H$ such that $f'(v) = h_1$ and $f'(u) = h_2$. Since there is no edge joining h_1 and h_2 there can be no edge between v and u. Since the choice of v and u was arbitrary, G has no edges.

Now let G be a graph without edges and H be any graph. Then any function $f_V : V(G) \to V(H)$ is a morphism in $\mathcal{G}$ since it automatically preserves incidence. Thus, the morphisms from G to H in $\mathcal{G}$ correspond exactly with those from $F(G)$ to $F(H)$ in $\mathcal{S}$. Hence, G is a full initial object in $\mathcal{G}$. □

Proposition 6.2. *The full terminal objects in the category $\mathcal{G}$ are the complete graphs.*

Proof: Let K_n be a complete graph and G be any other graph. Let $f_V : V(G) \to V(K_n)$ be any function. For $\{u,v\}$ an edge in G, define $f_E(\{u,v\}) = \{f(u), f(v)\}$. This definition is possible since there is an edge incident on every pair of vertices in K_n, and together f_V and f_E form a graph morphism from G to K_n. Since f_V can be chosen arbitrarily, K_n is a full terminal object.

Now let G be a full terminal object with distinct vertices u and v. Then there exists a non-trivial mapping f from the set $\{0,1\}$ to $V(G)$ such that $f(0) = u$ and $f(1) = v$. It is the image under the forgetful functor of a morphism from K_2, which is a single edge, to the vertices u and v of G . Hence, there is an edge between any pair of vertices of G, and therefore G is a complete graph. □

There is an interesting relationship between the graphs that are full terminal objects and those that are full initial objects. The *complement* of a graph G is the graph $\overline{G}$ which has the same vertex set as G but there is an edge incident on a pair of vertices in $\overline{G}$ if and only if there is no edge incident on the pair in G.

It follows immediately that a graph is a full terminal object if and only if its complement is a full initial object. The situation is made more interesting since a full initial object belongs to a coreflective subcategory and a full terminal object belongs to a reflective category.

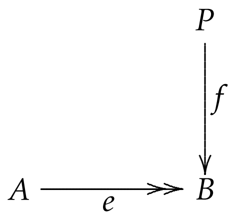
In terms of a parallel with the category of topological spaces, the graphs with no edges correspond to the discrete topological spaces on which every function is continuous, and the complete graphs correspond to the indiscrete topological spaces which have the property that every function to such a space is continuous.

The idea that there are a lot of morphisms from a graph without edges and to a complete graph will also be expressed in the next section.

7. Projective and injective objects in $\mathcal{G}$.

In the previous section we saw that a graph without edges is a full initial object and that we can therefore expect many morphisms to emanate from such an object. Here we further investigate the existence of such morphisms.

An object P in a category $\mathcal{C}$ is said to be *projective* if given the following diagram



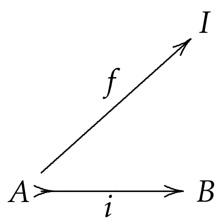
there exists a morphism $g : P \to A$ such that $f = e \circ g$.

Proposition 7.1. *The projective objects in $\mathcal{G}$ are the graphs without edges.*

Proof: Let P be a graph without edges, $e : G \twoheadrightarrow H$ be an epimorphism, and $f : P \to H$ be a morphism. For each vertex $v = f_V(v')$ in the image of f there exists a vertex u in $e_V^{\leftarrow}(v)$. Define $g_V(v') = u$. Thus $g : P \to G$ is a morphism, $f = e \circ g$, and hence P is projective.

Now suppose that P contains an edge $\{u, v\}$ and that $f : P \to H$ is a morphism such that $f_V(u) \neq f_V(v)$. Let G be the graph obtained from H by eliminating the edge $\{f_V(u), f_V(v)\}$. Define $e : G \twoheadrightarrow H$ to be the identity. Then e is an epimorphism because e_V is onto, but no morphism $g : P \to G$ to make P projective can exist because no composition $e \circ g$ can include the edge $\{f_V(u), f_V(v)\}$ in its image. □

An object I in the category $\mathcal{C}$ is said to be *injective* if given the following diagram



there exists a morphism $g : B \to I$ such that $f = g \circ i$.

Since projective and injective objects are dual concepts, we should not be surprised that their characterizations and proofs are "dual" in nature.

Proposition 7.2. *The injective objects in the category $\mathcal{G}$ are the complete graphs.*

Proof: Let I be a complete graph, let $f : H \to I$ be a homomorphism, and let $i : H \to G$ be a monomorphism. If $v_i = i_V(u_i), i = 1, 2$, define $g_V(v_i) = f_V(u_i)$. Then g_V is well defined since i_V is one-to-one. If the edge $v_1 v_2$ exists in G, then $g_E(v_1 v_2) = f(u_1) f(u_2)$ can be defined since I is complete. If $v \neq i_V(u)$ for any u in H but is adjacent to to a vertex v' such that $v' = i_V(u')$, define $g_V(v) = u'$. For vertices in G not adjacent to a vertex in the image of i, $g_V(u)$ can be defined arbitrarily so long as adjacency is preserved.

Then $g : G \to I$ is a homomorphism, $f = g \circ i$, and I is injective.

Now assume that L is a graph containing vertices u and v with no edge between them, and let $f : G \to L$ be a homomorphism such that vertices u' and v' exist in G with $f_V(u') = u$, $f_V(v') = v$. Let H be G with the edge $u'v'$ added. Then the embedding $i : G \rightarrowtail H$ is a monomorphism, but no homomorphism $G : H \to L$ can exist with $f = g \circ i$. □

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