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A Generalization of Placing Identical Items into Identical Bins

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Abstract

A common approach to counting the number of ways to place identical items into identical bins is by casework. In this article, an alternative approach is introduced and robust mathematical formulas are established to calculate the number of ways of placing arbitrary number of identical items into arbitrary number of identical bins. Firstly, single closed formulas for the cases of two and three bins are developed for arbitrary number of items. Secondly, a recursive formula for more than three bins is derived for arbitrary number of items. This recursive formula reduces the number of bins by one in each step until reaching the base case of three bins for which the closed formula derived in this paper can be applied. A Python program is implemented using the derived formulas that can count the number of ways for arbitrary bins and items.

1. Introduction

The purpose of this article is to explore alternative methods to find the number of ways to place items into bins, provided that both the items and bins are identical. Combinations with repetition are used to count the number of ways to place identical items into distinct bins. When both the items and bins are indistinguishable, the counting problem is formulated as a partition into at most k parts problem; see, for instance, Example 11 on page 454 in Rosen [7] (8th ed., 2019) or [1,4]. We will explore this in Section 1.2. Our pursuit of finding formulas for this problem should not be confused with the difficult partition of an integer.

The common method to count the number of ways to place identical items into identical bins is by casework, which is tedious, time-consuming, and sometimes not even feasible if the numbers of items and bins are large. It is well known that when counting partitions, no simple closed formula exists. This motivates our study and we shall present our formulas for this counting problem. In this introduction, we first recall the classical method of stars and bars, then present our main result for the case of identical bins through our alternative approach inspired by the method of stars and bars.

1.1. Combinations with repetition for distinguishable bins

Suppose you are hosting a birthday party. You want to know how many ways there are to arrange 8 identical balloons on 3 different tables. How would you solve this? The answer is: *combinations with repetition*.

Combinations with repetition, also called *stars and bars*, *stones and sticks*, or *balls and urns*, is a method that can be used to find the number of ways to put indistinguishable balls into distinguishable bins, e.g., give identical pieces of candy to different kids, and the like. William Feller, a famous mathematician in his well-known book [5] on probability, made it a widely used approach.

Let us first explain what combinations with repetition is and use it to help the host solve this problem.

Let us say that a star $*$ represents a balloon. We can arrange the 8 balloons in a row, like this:

$$********$$

We can think the distribution of the balloons among the 3 tables as inserting 2 dividers called bars $|$ into the row of balloons. For example, we put 2 balloons on the first table, 4 on the second, and 2 on the third. We can write this as $(2,4,2)$ and illustrate it as:

$$**|****|**$$

The first table has the balloons to the left of the first divider, the second table has the balloons between the first and the second dividers, and the third table has the balloons to the right of the second divider. If we want a table to have zero balloons, e.g., $(3,0,5)$, we can depict it as:

$$***||*****$$

So to determine the allotment of 8 balloons to 3 tables is the same as placing 2 dividers into the row of 8 balloons. This is also equal to arranging 10 symbols, 8 stars and 2 bars. The number of ways to do this is

$$\frac{10!}{8!\cdot 2!} = \binom{10}{2}.$$

So the number of ways to arrange 8 balloons on 3 different tables is $\binom{10}{2}$, which is 45.

In general, to distribute n items among k bins we need $k-1$ dividers in a row of n items. This is equal to positioning n stars and $k-1$ bars, which can be done in the following number of ways:

$$\frac{(n+k-1)!}{n!(k-1)!} = \binom{n+k-1}{k-1}.$$

1.2. Partitions into at most k parts

Now let us say that not only the balloons are the same, but also the tables are the same. How many such arrangements are possible? Our situation is the same as counting the number of ways to place n identical items into k identical bins. This is also the same as counting the number of ways to partition a positive integer n into at most k parts, which is denoted by $p_k(n)$. "At most k parts" corresponds to some of the bins being possibly empty. A partition of n is a method of breaking down n into a sum of smaller, positive integers. The order of the summands does not matter. For example, for $n = 8$ and $k = 2$, the partitions of 8 into at most 2 parts are $8, 7+1, 6+2, 5+3, 4+4$ for a total of 5 partitions. So $p_2(8) = 5$.

In considering a specific case of $p_k(n)$, we will express the partition

$$n = n_1 + n_2 + \cdots + n_k,$$

where $n_1 \geq n_2 \geq \cdots \geq n_k$, as $(n_1, n_2, \cdots, n_k)$. Then we can express the different partitions of $p_2(8)$ as $(8,0),(7,1),(6,2),(5,3),(4,4)$.

Table 1.1: Values for $p_k(n)$

n \ k	1	2	3	4	5	6	7	8	9
1	1								
2	1	2							
3	1	2	3						
4	1	3	4	5					
5	1	3	5	6	7				
6	1	4	7	9	10	11			
7	1	4	8	11	13	14	15		
8	1	5	10	15	18	20	21	22	
9	1	5	12	18	23	26	28	29	30

Note $p_k(n) = p_n(n)$ when $k \geq n$, i.e., all blank spaces are equal to the leftmost value that is closest to it.

There is also a more general partition of an integer n, $p(n)$, that has no restrictions except that the order of the summands does not matter. For example, for $n = 5$, the partitions of 5 are $5, 4+1, 3+2, 3+1+1, 2+2+1, 2+1+1+1, 1+1+1+1$ for a total of 7 partitions. So $p(5) = 7$.

1.3. New method for indistinguishable bins

When both the balloons and the tables are the same, you could use casework. For example, the classical and recent books [2, 5–8] provide some examples by casework where the number of items is small and the number of bins is only up to 3. In this article, we will explore other methods to solve problems involving that both the items *and* the bins, tables, etc. are indistinguishable. We also permit the bins to have 0 items.

A trivial case is $n = 0$ for which $p_k(0) = 1$ for any integer $k \geq 1$. Another trivial case is $p_1(n) = 1$ for any integer $n \geq 0$.

We first state the main result, providing the closed and recursive formulas for counting the number of ways to place identical items into identical bins.

Theorem (Main Result)**.** Let $p_k(n)$ denote the total number of ways to place n indistinguishable items into k indistinguishable bins for given nonnegative integers n and k. Then

(1) For $k = 2$,

$$p_2(n) = \left\lfloor \frac{n}{2} \right\rfloor + 1. \tag{1.1}$$

(2) For $k = 3$,

$$p_3(n) = \frac{\binom{n+2}{2} - \left\lceil \frac{n \bmod 3}{2} \right\rceil \times 2 + \left\lfloor \frac{n}{2} \right\rfloor \times 3 + 5}{3!}. \tag{1.2}$$

(3) For $k \geq 4$,

$$p_k(n) = \sum_{l=0}^{\lfloor \frac{n}{k} \rfloor} p_{k-1}(n - kl). \tag{1.3}$$

Here $\lfloor\,\rfloor$ denotes the floor function, $\lceil\,\rceil$ denotes the ceiling function, and a mod b denotes the remainder of the Euclidean division of a by b.

We will prove (1.1) in Section 2, (1.2) in Section 4, and (1.3) in Sections 5-6.

We note that the two formulas (1.1) and (1.2) are single closed formulas for two and three identical bins respectively. The formula (1.3) is a recursive formula for more than three identical bins. All three powerful formulas work for any given nonnegative number n of bins. When $k \geq 4$, using formula (1.3) reduces to the case of $k-1$ bins, then using (1.3) again reduces further to $k-2$ bins, and we continue this way until it reduces to 3 bins where the single closed formula (1.2) can be used.

Our approach to obtain these formulas roughly consists of two steps: (1) counting the number of ways to place the identical items into distinct bins by the method of stars and bars, and (2) subtracting the repeats when the bins are also identical. The challenge is to find out a single formula that works for arbitrary numbers of items and bins.

For the easy case of two identical bins, i.e., $k = 2$, we start with two real examples with $n = 8, 9$, then generalize to obtain the formula (1.1).

When we have three identical bins, it is not possible to use just casework to derive a formula for an arbitrary number n. Instead, we first count the number of ways to place identical items into 3 bins by the method of stars and bars if the bins are distinct. Since the bins are in fact indistinguishable, there are repeats in the counting via the stars and bars method, thus we need to eliminate those repeats. In order to show how to eliminate the repeats correctly, we use two real examples of eight and nine items. Then we consider two general cases: the number of items n is (a) not a multiple of three, and (b) a multiple of three. After deriving the formula for each of the two cases, we combine them into one single closed formula (1.2) using the modulo function $a \bmod b$.

For the case of more than three bins, we first use four bins to explain our idea. If we place l items into the first bin for $l = 0, 1, 2, \ldots, \left\lfloor \frac{n}{4} \right\rfloor$, then we shall have $n - 4l$ items to place into the remaining three bins freely, which can be counted by the previous formula (1.2) for $k = 3$. This idea can be generalized to the case of $k > 4$ as follows. If we place l items into the first bin for $l = 0, 1, 2, \ldots, \left\lfloor \frac{n}{k} \right\rfloor$, then we shall have $n - kl$ items to place into the remaining $k - 1$ bins freely, which leads to the recursive formula (1.3).

In the rest of the article, we consider the case of two bins in Section 2, a recurrence relation for $p_k(n)$ in Section 3, the case of three bins in Section 4, the case of four bins in Section 5, the general case of k bins with $k > 4$ in Section 6, an application in Section 7, and a Python program for arbitrary values of bins and items in Section 8.

2. Two Identical Bins: $k = 2$

In this section, we consider the easy case of two bins, i.e., $k = 2$. Let us start with two examples. Consider the first example:

Example 2.1: Let us consider our birthday party again. Suppose this time you want to arrange 8 identical balloons on 2 identical tables. How many ways are there?

Solution: We can solve this by casework. Since the tables are indistinguishable, only the number of balloons on each table matters. We list out the possibilities:

$$(8,0),(7,1),(6,2),(5,3),(4,4).$$

There are a total of 5 possibilities.

The second example is the following:

Example 2.2: If you want to put 9 identical balloons on 2 identical tables, how many ways are there?

Solution: Solving this problem using casework gives us the possibilities:

$$(9,0),(8,1),(7,2),(6,3),(5,4).$$

There are in total 5 possibilities.

Now let us use our new approach to consider these two examples. For the first example, if the balloons are identical and the tables are distinguishable, then the total number of ways is $\binom{8+2-1}{2-1} = \binom{9}{1} = 9$. But since both are indistinguishable, we need to eliminate the repeats. For this purpose, first we have to find the number of ways to arrange the balloons such that both tables have the same number. There is one possibility where both tables get the same number of balloons. Now we have to find the ways that each table gets a distinct number of of balloons. We have $9-1=8$ ways. But since the tables are also identical, we need to divide by the number of ways we can order them, which is $2! = 2$. So the total number of ways is $\frac{8}{2!} = 4$. Now we need to add the one possibility in which both tables have the same number of balloons because we subtracted it. Therefore, the total number of ways to arrange 8 identical balloons on 2 identical tables is $4+1=5$.

Using the same logic for the second example, if the tables are distinguishable but the balloons are not, there are a total of $\binom{9+2-1}{2-1} = \binom{10}{1} = 10$ ways. But since both are identical, we need to remove the repeats. Here, all the possibilities are arrangements of which each table has a distinct number of balloons. We have 10 ways. However, since the tables are also identical, we need to divide by the number of ways we can order them, which is $2! = 2$. Therefore, the total number of ways to arrange 9 identical balloons on 2 identical tables is $\frac{10}{2!} = 5$.

We now generalize the above examples to general n balloons on 2 tables. If the tables are distinguishable, but the balloons are not, there are a total of

$$\binom{n+2-1}{2-1} = \binom{n+1}{1} = n+1$$

ways. But since both the balloons and tables are indistinguishable, we need to subtract the repeats. We will consider two cases:

(a) The first case is that n is a multiple of 2 as in Example 2.1. There is one possibility where both tables get the same number of balloons. There remain, as we noticed before, $n+1-1=n$ different possibilities. Each possibility has 2 orderings. So the total number of ways that each identical table gets a distinct number of balloons is $\frac{n}{2!} = \frac{n}{2}$. Hence, the total number of ways to arrange n identical balloons on 2 identical tables is $\frac{n}{2}+1$.

(b) The second case is that n is not a multiple of 2 as in Example 2.2. Here, all the possibilities are arrangements of which each table has a distinct number of balloons. We have $n+1$ ways. However, since the tables are identical, we need to divide by the number of ways we can order them, which is $2! = 2$. Therefore, the total number of ways to arrange n balloons on 2 tables is $\frac{n+1}{2} = \frac{n-1}{2}+1$.

Combining these two cases together, we can get the single formula $p_2(k) = \left\lfloor \frac{n}{2} \right\rfloor + 1$ which is (1.1).

Below we give another proof using the generalized pigeonhole principle. First, we introduce a lemma:

Lemma 2.1. *For any $k \geq 0$,*

(1) if $n = 2k$, then $\left\lceil \frac{n}{2} \right\rceil = \left\lfloor \frac{n}{2} \right\rfloor$.

(2) if $n = 2k+1$, then $\left\lceil \frac{n}{2} \right\rceil = \left\lfloor \frac{n}{2} \right\rfloor + 1$.

Proof. (1) $\left\lceil \frac{n}{2} \right\rceil = \left\lceil \frac{2k}{2} \right\rceil = \lceil k \rceil = k = \left\lfloor \frac{2k}{2} \right\rfloor = \left\lfloor \frac{n}{2} \right\rfloor$.
(2) $\left\lceil \frac{n}{2} \right\rceil = \left\lceil \frac{2k+1}{2} \right\rceil = \left\lceil k + \frac{1}{2} \right\rceil = k + 1 = \left\lfloor k + \frac{1}{2} \right\rfloor + 1 = \left\lfloor \frac{2k+1}{2} \right\rfloor + 1 = \left\lfloor \frac{n}{2} \right\rfloor + 1$. □

Now we can prove Equation (1.1):

Proof of Equation (1.1). Since there are two bins, by the generalized pigeonhole principle, one of the bins must have at least $\lceil \frac{n}{2} \rceil$ items. Therefore,

$$p_2(n) = \left| \left\{ \left(\left\lceil \frac{n}{2} \right\rceil, n - \left\lceil \frac{n}{2} \right\rceil \right), \left(\left\lceil \frac{n}{2} \right\rceil + 1, n - \left\lceil \frac{n}{2} \right\rceil - 1 \right), \cdots, (n, 0) \right\} \right|.$$

There are two cases:
<u>Case 1:</u> n is even. Then $n = 2k, k \in \mathbb{Z}$.
Then

$$\begin{aligned}
p_2(n) &= \left| \left\{ \left(\left\lceil \frac{2k}{2} \right\rceil, n - \left\lceil \frac{2k}{2} \right\rceil \right), \left(\left\lceil \frac{2k}{2} \right\rceil + 1, n - \left\lceil \frac{2k}{2} \right\rceil - 1 \right), \cdots, (n, 0) \right\} \right| \\
&= |\{(k, n-k), (k+1, n-k-1), \cdots, (n, 0)\}| \\
&= n - k + 1 \\
&= 2k - k + 1 = k + 1 \\
&= \left\lfloor \frac{n}{2} \right\rfloor + 1 \text{ by Lemma 2.1 (1).}
\end{aligned}$$

<u>Case 2:</u> n is odd. Then $n = 2k+1, k \in \mathbb{Z}$.
Then

$$\begin{aligned}
p_2(n) &= \left| \left\{ \left(\left\lceil \frac{2k+1}{2} \right\rceil, n - \left\lceil \frac{2k+1}{2} \right\rceil \right), \left(\left\lceil \frac{2k+1}{2} \right\rceil + 1, n - \left\lceil \frac{2k+1}{2} \right\rceil - 1 \right), \cdots, (n, 0) \right\} \right| \\
&= |\{(k+1, n-k-1), (k+2, n-k-2), \cdots, (n, 0)\}| \\
&= n - k \\
&= 2k + 1 - k = k + 1 \\
&= \left\lfloor \frac{n}{2} \right\rfloor + 1 \text{ by Lemma 2.1 (2).}
\end{aligned}$$

Thus, $p_2(n) = \left\lfloor \frac{n}{2} \right\rfloor + 1$. □

3. Recurrence relation for $p_k(n)$

We will use the following lemma in the section below:

Lemma 3.1. $p_k(n) = p_k(n-k) + p_{k-1}(n)$.

Proof. We will prove this relation by showing both sides count the same thing by counting two different ways. The left side of the equation is the number of ways to place n items into k bins.

The right side counts the same thing as follows:
(1) The first term is the total number of ways to place n items such that no bins are empty, namely, each bin has at least one item. Thus we can place $n-k$ items into k bins.
(2) The second term is the total number of ways such that at least one bin is empty. Thus we can place n items into $k-1$ bins.

Thus, the relation holds. $\square$

4. Three Identical Bins: $k = 3$

In this section we consider three identical bins. We also start with two special examples.
Example 4.1: Let us consider our birthday party again. Suppose this time you want to arrange 8 identical balloons on 3 identical tables. How many ways are there?
Solution: Using casework we see that there are 10 possibilities:

$$(8,0,0),(7,1,0),(6,2,0),(6,1,1),(5,3,0),(5,2,1),(4,4,0),(4,3,1),(4,2,2),(3,3,2).$$

Example 4.2: If you want to put 9 identical balloons on 3 identical tables, how many ways are there?
Solution: Solving by casework, we see that there are 12 possibilities:

$$(9,0,0),(8,1,0),(7,2,0),(7,1,1),(6,3,0),(6,2,1),$$

$$(5,4,0),(5,3,1),(5,2,2),(4,4,1),(4,3,2),(3,3,3).$$

When the number of items is large, it is difficult to count the number of ways by casework. Thus a new approach is needed.

4.1. Method 1: Using combinations with repetition and eliminating repeats

Now we present our approach.

For Example 4.1, if the balloons are identical and the tables are distinguishable, then the total number of ways is $\binom{8+3-1}{3-1} = \binom{10}{2} = 45$. But since both are indistinguishable, we need to eliminate the repeats, of which there are two kinds: exactly two tables have the same number, and no tables have the same number of balloons. We do the following:

Step 1: Count how many ways to arrange the balloons such that two tables have the same number. There are 5 different possibilities: (0,0,8), (1,1,6), (2,2,4), (3,3,2), (4,4,0), each of which has $\frac{3!}{2!} = 3$ orderings. So the total number of ways is $3 \times 5 = 15$.

Step 2: Count the number of ways to arrange the balloons such that no table have the same number. We have $45 - 15 = 30$ ways. But since the tables are also identical, we need to divide by the number of ways we can order them, which is $3! = 6$. So the total number of ways is $\frac{30}{3!} = 5$.

Step 3: Now we need to add the 5 different possibilities in which exactly two tables have the same number of balloons, because we subtracted them. Therefore, the total number of ways to arrange 8 identical balloons on 3 identical tables is $5 + 5 = 10$.

For Example 4.2, we can use the same logic as in Example 4.1. If the tables are distinguishable but the tables are not, there are a total of $\binom{9+3-1}{3-1} = \binom{11}{2} = 55$. But since both are identical, we need to take off the repeats, of which there are three kinds: all three tables have the same number, exactly two tables have the same number, and no tables have the same number of balloons. We do the following:

Step 1: For this problem, we have a possibility where all three tables get the same number of balloons. There is only one way, that is, (3,3,3).

Step 2: Count how many ways to arrange the balloons such that exactly two tables have the same number. There are 4 different ways to arrange the balloons such that only two tables have the same number (note that (3,3,3) does not belong to this category since all three tables have the same number). Each of these possibilities has $\frac{3!}{2!} = 3$ orderings. So the total number of ways is $3 \times 4 = 12$.

Step 3: Count the number of ways to arrange the balloons such that no table have the same number. We have $55 - 12 = 43$, and there is only one way where all three tables get the same number of balloons. So this leaves us with all the ways that the possibilities have all three numbers distinct, which is $43 - 1 = 42$. But since the tables are also indistinguishable, we need to divide by the number of ways we can order them, which is $3! = 6$. Therefore the total number of ways to arrange the balloons such that each table gets a different number is $\frac{42}{3!} = 7$.

Step 4: Now we need to add the 4 different possibilities in which exactly two tables have the same number of balloons and the one possibility where all three tables get the same number since we subtracted them. Thus, the total number of ways to arrange 9 identical balloons on 3 identical tables is $7 + 4 + 1 = 12$.

We now can generalize the idea to n balloons on 3 tables. We will consider two cases: n is or is not a multiple of 3, that is, $n \equiv 0 \pmod 3$ or $n \equiv 1, 2 \pmod 3$.

The first case is that n is not a multiple of 3 as in Example 4.1. Let us say n is the number of balloons.

Lemma 4.1. *For $n \equiv 1, 2 \pmod 3$,*

$$p_3(n) = \frac{\binom{n+2}{2} - \left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right) \times 3}{3!} + \left\lfloor \frac{n}{2} \right\rfloor + 1.$$

Proof. If the tables are distinguishable, but the balloons are not, then using the formula for stars and bars, we get $\binom{n+3-1}{3-1} = \binom{n+2}{2}$. But since both the balloons and tables are indistinguishable, we need to subtract the repeats, of which there are two kinds: exactly two tables have the same number, and no tables have the same number of balloons. We do the following:

Step 1: Count how many ways to arrange the balloons such that two tables have the same number of balloons. There are $\left\lfloor \frac{n}{2} \right\rfloor + 1$ different possibilities. Each of these has 3 orderings since there are 3 ways to place the last balloon. So the total number of ways is $(\left\lfloor \frac{n}{2} \right\rfloor + 1) \times 3$.

Step 2: Count the number of ways to arrange the balloons such that no tables have the same number. We have

$$\binom{n+2}{2} - \left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right) \times 3$$

ways. But since the tables are also identical, we need to divide by the number of ways we can order them, which is $3! = 6$. So the total number of ways to arrange the identical balloons on identical tables such that each table has a different number is

$$\frac{\binom{n+2}{2} - \left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right) \times 3}{3!}.$$

Step 3: Now we need to add the $\left\lfloor \frac{n}{2} \right\rfloor + 1$ possibilities in which exactly two tables have the same number of balloons because we subtracted them.

Therefore, the total number of ways to arrange n identical balloons on 3 identical tables is given by

$$p_3(n) = \frac{\binom{n+2}{2} - \left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right) \times 3}{3!} + \left\lfloor \frac{n}{2} \right\rfloor + 1$$

when $n \equiv 1, 2 \pmod 3$. □

The second case is that n is a multiple of 3.

Lemma 4.2. *For $n \equiv 0 \pmod 3$,*

$$p_3(n) = \frac{\binom{n+2}{2} - \left\lfloor \frac{n}{2} \right\rfloor \times 3 - 1}{3!} + \left\lfloor \frac{n}{2} \right\rfloor + 1.$$

Proof. We can use the same logic as in the first case. If the tables are distinguishable, but the balloons are not, there are a total of $\binom{n+3-1}{3-1} = \binom{n+2}{2}$. Since both the balloons and tables are indistinguishable, we need to subtract the repeats, of which there are three kinds: all three tables have the same number, exactly two tables have the same number, and no tables have the same number of balloons. We do the following:

Step 1: Since n is a multiple of 3, we also have a possibility where all three tables get the same number of balloons. There is only one ordering for this case.

Step 2: Count how many ways to arrange the balloons such that exactly two tables have the same number of balloons. There are, as we noticed before, $\left\lfloor \frac{n}{2} \right\rfloor$ different possibilities. Each possibility has 3 orderings. So the total number of ways is $\left\lfloor \frac{n}{2} \right\rfloor \times 3$.

Step 3: Subtracting the sum of Steps 1 and 2 from $\binom{n+2}{2}$ leaves us with all the ways with all three numbers distinct, and the number of ways is

$$\binom{n+2}{2} - \left\lfloor \frac{n}{2} \right\rfloor \times 3 - 1.$$

But since the tables are also indistinguishable, we need to divide by the number of ways we can order them, which is $3! = 6$. So the total number of ways to arrange the identical balloons on identical tables such that no tables have the same number of balloons is

$$\frac{\binom{n+2}{2} - \left\lfloor \frac{n}{2} \right\rfloor \times 3 - 1}{3!}.$$

Step 4: Now we need to add the $\left\lfloor \frac{n}{2} \right\rfloor$ possibilities in which exactly two tables have the same number of balloons and the one possibility where all three tables get the same number since we subtracted them. Therefore, the total number of ways to arrange n identical balloons on 3 identical tables is

$$p_3(n) = \frac{\binom{n+2}{2} - \left\lfloor \frac{n}{2} \right\rfloor \times 3 - 1}{3!} + \left\lfloor \frac{n}{2} \right\rfloor + 1$$

when $n \equiv 0 \pmod 3$. □

Table 4.1 below illustrates the proofs of the two lemmas.

We can combine Lemmas 4.1 and 4.2 into one single formula:

$$p_3(n) = \frac{\binom{n+2}{2} - \left(\left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{n \bmod 3}{2} \right\rceil \right) \times 3 + \left\lceil \frac{n \bmod 3}{2} \right\rceil - 1}{3!} + \left\lfloor \frac{n}{2} \right\rfloor + 1.$$

Simplifying, we get

$$p_3(n) = \frac{\binom{n+2}{2} - \left\lceil \frac{n \bmod 3}{2} \right\rceil \times 2 + \left\lfloor \frac{n}{2} \right\rfloor \times 3 + 5}{3!},$$

Table 4.1: Obtaining a formula for $p_3(n)$:
number of ways to place n indistinguishable balloons on 3 indistinguishable tables

$k=3$ tables		$n \equiv 0 \pmod 3$	$n \equiv 1,2 \pmod 3$
Distinguishable Tables		$\binom{n+3-1}{3-1} = \binom{n+2}{2}$	$\binom{n+3-1}{3-1} = \binom{n+2}{2}$
Indistinguishable	All three tables have the same number of balloons	1	Not feasible
Tables	Exactly two of the tables have the same number of balloons	$\left\lfloor \frac{n}{2} \right\rfloor + 1 - 1 = \left\lfloor \frac{n}{2} \right\rfloor$	$p_2(n) = \left\lfloor \frac{n}{2} \right\rfloor + 1$
	No tables have the same number of balloons	$\frac{\binom{n+2}{2}-1-\lfloor \frac{n}{2} \rfloor \times 3}{3!}$	$\frac{\binom{n+2}{2}-(\lfloor \frac{n}{2} \rfloor+1)\times 3}{3!}$
Total number of ways		$\frac{\binom{n+2}{2}-\lfloor \frac{n}{2} \rfloor \times 3-1}{3!} + \left\lfloor \frac{n}{2} \right\rfloor + 1$	$\frac{\binom{n+2}{2}-(\lfloor \frac{n}{2} \rfloor+1)\times 3}{3!} + \left\lfloor \frac{n}{2} \right\rfloor + 1$

which is the formula (1.2).

4.2. Method 2: Using recurrence relation 3.1

Another approach is to use the recurrence relation 3.1 from Section 3:

$$p_k(n) = p_k(n-k) + p_{k-1}(n), \quad n \geq k.$$

For $k = 3$,

$$p_3(n) = p_3(n-3) + p_2(n), \quad n \geq 3. \tag{4.1}$$

So we have

$$\begin{aligned} p_3(n) &= p_3(n-3) + p_2(n) \\ &= p_3(A) + p_2(n), \qquad A \geq 0,\ n = A+3, \\ &= \begin{cases} 1 + p_2(n), & A = 0, \\ 1 + p_2(n), & A = 1, \\ 2 + p_2(n), & A = 2, \\ p_3(B) + p_2(A) + p_2(n), & A = 3,\ B = A-3. \end{cases} \end{aligned}$$

We also have

$$p_3(0) = 1, \quad p_3(1) = 1, \quad p_3(2) = 2.$$

Let us write out what $p_3(n)$ is using (4.1) for the first few values of $n \geq 3$:

$$p_3(n) = \begin{cases} 1 + p_2(3), & n = 3, \\ 1 + p_2(4), & n = 4, \\ 2 + p_2(5), & n = 5, \\ 1 + p_2(3) + p_2(6), & n = 6, \\ 1 + p_2(4) + p_2(7), & n = 7, \\ 2 + p_2(5) + p_2(8), & n = 8, \\ 1 + p_2(3) + p_2(6) + p_2(9), & n = 9, \\ 1 + p_2(4) + p_2(7) + p_2(10), & n = 10, \\ 2 + p_2(5) + p_2(8) + p_2(11), & n = 11, \\ 1 + p_2(3) + p_2(6) + p_2(9) + p_2(12), & n = 12, \\ 1 + p_2(4) + p_2(7) + p_2(10) + p_2(13), & n = 13, \\ 2 + p_2(5) + p_2(8) + p_2(11) + p_2(14), & n = 14. \end{cases}$$

Using an inductive argument, we can obtain another formula:

$$p_3(n) = p_3(n \bmod 3) + \sum_{i=1}^{m} p_2(3i + l), \tag{4.2}$$

where $n = 3m + l,\ l = 0, 1, 2,\ m = \left\lfloor \frac{n}{3} \right\rfloor$.

4.3. An identity for $p_3(n)$

Because we count the same thing in Sections 4.1 and 4.2, we have established the following identity:

$$\frac{\binom{n+2}{2} - \left\lceil \frac{n \bmod 3}{2} \right\rceil \times 2 + \left\lfloor \frac{n}{2} \right\rfloor \times 3 + 5}{3!} = p_3(n \bmod 3) + \sum_{i=1}^{m} p_2(3i + l). \tag{4.3}$$

5. Four Identical Bins: $k = 4$

In this section we consider the case of four identical bins. As an example, let us continue on our birthday party.

Example 5.1: Suppose there are 9 indistinguishable balloons and 4 indistinguishable tables. How many ways to arrange them?

Solution: The casework shows 18 possibilities:

$$(0,9,0,0),(0,8,1,0),(0,7,2,0),(0,7,1,1),(0,6,3,0),(0,6,2,1),$$

$$(0,5,4,0),(0,5,3,1),(0,5,2,2),(0,4,4,1),(0,4,3,2),(0,3,3,3),$$

$$(1,6,1,1),(1,5,2,1),(1,4,3,1),(1,4,2,2),(1,3,3,2),(2,3,2,2).$$

One can imagine that if the number of balloons is large, the casework takes forever. Thus we need an efficient way to accomplish it.

We now introduce our approach. We have to decide how many balloons to put on the first table. Once this is decided, the problem reduces to the 3-table problem in Section 4.

Case 1: If the first table gets zero balloons, we have 9 balloons left to put on the remaining 3 tables, which we know there are 12 arrangements.

Case 2: If the first table gets one balloon, we have 8 balloons left to put on the remaining 3 tables. This leads to the arrangement of 8 balloons on 3 tables which we discussed earlier. However, we now cannot have a table with zero balloons because otherwise it will repeat the case when the first table gets zero balloons, which means that each of the three tables should have at least one balloon. So let us first put one balloon on each of the rest three tables. We have $9-1-1\times 3 = 9-4 = 5$ balloons left which we can put on the 3 tables without restrictions. There are 5 ways using the formula (1.2).

Case 3: If the first table gets two balloons, we have 7 balloons left to put on the remaining 3 tables. But now we cannot have a table with 0 or 1 balloon because otherwise it will repeat the cases when the first table gets 0 or 1 balloon, which means that each of the three tables should have at least two balloons. Let us first put 2 balloons on each of the remaining 3 tables. So we have $9-2-2\times 3 = 9-8 = 1$ balloon left, which we can put on 3 tables one way using the formula (1.2).

Case 4: If the first table gets three balloons, we have 6 balloons left to put on the remaining 3 tables. But now we cannot have a table with 0, 1 or 2 balloons because otherwise it will repeat the cases when the first table gets 0, 1 or 2 balloons, which means that each of the three tables should have at least three balloons. Since we have only 6 balloons for the three tables, this is impossible, which means we should stop when the first table gets 2 balloons.

By adding these four cases, we get $12+5+1 = 18$ ways to distribute 9 balloons on 4 tables.

We now generalize the approach to n balloons on 4 tables. Let us say n is the total number of balloons and l is the number of balloons on the first of the four tables. Then, as in the above argument, there are $n-l$ balloons for the remaining 3 tables. To avoid repeats, each of the 3 tables cannot have $0,1,\cdots,$ or $l-1$ balloons, which means each table should have at least l balloons. After we put l balloons on the 3 tables, we have $n-l-3l=n-4l$ balloons left to put on the three tables randomly. Thus it has $p_3(n-4l)$ ways using the formula (1.2). This can be explained in more detail as follows.

If $l=0$, then the number of ways is $p_3(n-l-3l)=p_3(n-4l)=p_3(n)$.

If $l=1$, then the number of ways is $p_3(n-l-3l)=p_3(n-4l)=p_3(n-4)$.

If $l=2$, then the number of ways is $p_3(n-l-3l)=p_3(n-4l)=p_3(n-8)$.

We can continue this way until the maximum number is reached for l, explained as follows. First, $n-4l$ has to be bigger than or equal to 0 since it cannot be negative, i.e., $n-4l\geq 0$. That means $l\leq \frac{n}{4}$. The biggest integer l that is less than or equal to $\frac{n}{4}$ is $\left\lfloor \frac{n}{4}\right\rfloor$. So the total number of ways to put n identical balloons on 4 identical tables is

$$\sum_{l=0}^{\lfloor \frac{n}{4}\rfloor} p_3(n-4l),$$

which is the formula (1.3) for $k=4$.

6. More Identical Bins: $k>4$

In this section we generalize the procedure in the previous section for four bins to more bins.

Let us say n is the number of identical balloons, k is the number of identical tables, and l is the number of balloons on the first table. Then we have $n-l-(k-1)l=n-kl$ balloons left to put on the remaining $k-1$ tables randomly, thus the number of ways of configurations is $p_{k-1}(n-kl)$. Here l starts from 0, then 1, and continue until $\left\lfloor \frac{n}{k}\right\rfloor$ since $n-kl$ has to be nonnegative.

We need to add them up to arrange n identical balloons on k identical tables. Therefore, the total number of ways to put n balloons on k tables when $k\geq 4$ is

$$\sum_{l=0}^{\lfloor \frac{n}{k}\rfloor} p_{k-1}(n-kl),$$

which is the formula (1.3) for general $k>4$. We remark that the formula (1.3) is a recursive formula, and it works for any integer $n>0$ and any integer $k\geq 4$, via the single closed formula (1.2) for base case $k=3$.

7. Applications

The general formula (1.3) we developed above can come very handy when either n or k or both are large, since casework would not be efficient. We illustrate this in the following two examples.

Example 7.1: In [7], Example 11 on page 454 counts the case of 6 identical books into 4 identical boxes. The author uses casework to conclude that there are 9 ways to pack the books into the boxes. We can use our formula (1.3) for $k = 4$ and the closed formula (1.2) for $k = 3$ to obtain the same answer easily as the following:

$$
\begin{aligned}
p_4(6) &= \sum_{l=0}^{\lfloor \frac{6}{4} \rfloor} p_{4-1}(6-4l) \\
&= \sum_{l=0}^{1} p_3(6-4l) \\
&= p_3(6) + p_3(2) = 7 + 2 = 9.
\end{aligned}
$$

As aforementioned, when the numbers of items and bins are large, it is very difficult to count by casework, but our formulas are robust.

Example 7.2: An online store has 20 identical books to pack. They have 4 identical boxes. How many ways can they distribute the books to the boxes?

Solution: Using the formula (1.3) for $k = 4$ and then the formula (1.2) for $k = 3$, we have

$$
\begin{aligned}
p_4(20) &= \sum_{l=0}^{\lfloor \frac{20}{4} \rfloor} p_{4-1}(20-4l) \\
&= \sum_{l=0}^{5} p_3(20-4l) \\
&= p_3(20) + p_3(16) + p_3(12) + p_3(8) + p_3(4) + p_3(5) \\
&= 44 + 30 + 19 + 10 + 4 + 1 = 108.
\end{aligned}
$$

Therefore, there are 108 ways to distribute 20 identical books to 4 identical boxes.

Again one can imagine that finding out correctly the number of ways by casework for the 108 cases in this example is not easy, but the above calculation using the formula (1.3) and then the formula (1.2) is effective.

For even larger n and k, we can use a computer program to find the number of ways based on the recursive formula (1.2); see the next section for an example.

8. Python Program

In this section, a Python program is presented that calculates the number of ways for arbitrary values of n and k. This program contains two functions (one original function and the main function) and uses recursion. The function starsBars, calculates the number of ways for three bins or less.

```
'''
The user gives the values for n and k.
n: nonnegative integer number of identical ITEMS
k: nonnegative integer number of identical BINS
'''

import math as m

def starsBars(n, k):
    if k == 0 or k == 1: #0 or 1 identical BINS
        return 1
    if k == 2: #2 identical BINS
        result = m.floor(n / 2) + 1
        return result
    elif k == 3: #3 identical BINS
        result = (m.comb(n + 2, 2) - (m.floor(n / 2)
                + m.ceil((n % 3) / 2)) * 3
                + m.ceil((n % 3) / 2) - 1) / m.factorial(3)
                + m.floor(n / 2) + 1
        return result
    elif k >= 4: #4 or more identical BINS
        sum = 0
        for l in range(0, m.floor(n / k) + 1):
            #Recursive formula
            sum += starsBars(n - (k * l), k - 1)
        return sum

def main():
    n = int(input('Enter number of identical ITEMS: '))
    k = int(input('Enter number of identical BINS: '))
    print('Number of ways to distribute n identical items
            to k different bins:', int(starsBars(n, k)))

if __name__ == '__main__':
    main()
```

Consider the case of $n = 100$ identical items and $k = 20$ identical bins. We can use the above Python program to obtain that the number of ways is 97132873.

```
Enter number of identical ITEMS: 100
Enter number of identical BINS: 20
Number of ways to distribute n identical items
                 to k different bins: 97132873
```

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