Pittsburgh Interdisciplinary Mathematics Review
Volume 2, December 2024, Pages 138–141
https://doi.org/10.5195/pimr.2024.45

Beach Math

Ryder Pham, Evan Hyzer, and Anastasiia Rudenko

What is Beach Math?

You're on the beach. The water's too chilly for a swim, and the sand is blazing outside of your trusty umbrella and towel. To top it off, you forgot your charger, and your novel is sitting at home because you thought you wouldn't need it. Your friends have wandered off, so you are left with nothing but your thoughts. What else is there to do but think about that math problem you saw in PIMR a while ago? That's where Beach Math comes in—bringing you challenging yet relaxing problems to keep your vacation boredom at bay. We hope you enjoy!

Problems

Calm Waves (Easy)

Classical. Alice and Bob are playing a game on a chessboard that is infinite going right and down but is bounded on the left and top. There is a single piece that is placed in some square on the board before the game. Alice and Bob take turns making valid moves, moving the piece either left or up any number of spaces, keeping the piece on the board. The first player who does not have a valid move is the loser. For which starting positions can Alice and Bob each force a win?

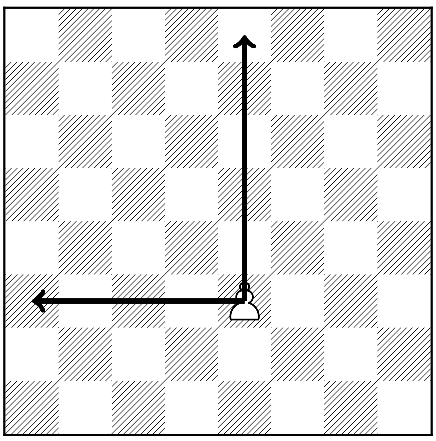

Figure 1: Valid Moves

Rough Waters (Medium)

There are n coins in a heap from which two thinkers take turns in claiming some number of coins. They are allowed to take any number of coins that is different from the number of coins taken on the previous turn. The first thinker is forbidden from taking all the coins on their first turn. The thinker who cannot take any coins is the loser. Which one of the two thinkers will win for which n and what strategy should they take?

Tsunami (Hard)

Let P_0 be the point at the bottom-left corner of a square of side length 2 centered at the origin and let P_1, P_2, P_3, P_4 be the midpoints of edges of the square starting from the left-most edge and going clockwise, as in the figure. Define P_n inductively to be the midpoint of segment $P_{n-4}P_{n-5}$. What is $\lim_{n\to\infty} P_n$?

A similar problem appears in James Stewart's Calculus*, 5 ed., Chapter 11 Problems Plus #18.*

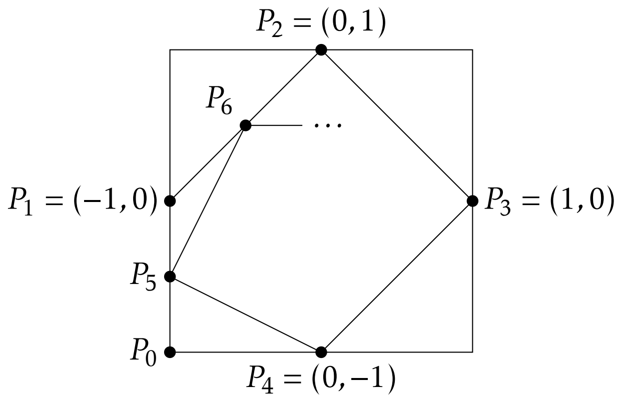

Figure 2: Tsunami

Solutions

Calm Waves (Easy)

Note that if the piece is in the top left, Alice automatically loses. Also notice that if the piece is ever on the diagonal, any move by a player can be matched by the other player to keep the piece on the diagonal, eventually allowing only the other player to move to the top left. Thus, the player that puts the piece on the diagonal will win. Therefore Bob wins if the piece is placed on the diagonal and Alice wins everywhere else.

Remark 1. *What happens if we allow diagonal moves to the upper-left? What about if we add moves to the upper-right or lower-left? What about knight's moves?*

Remark 2. *This game is an example of an* impartial *game since any valid move for Alice is also a valid move for Bob. It is also an example of a* combinatorial *game since players take turns and there is no hidden information.*

In the theory of impartial games, we call positions where the piece is on the diagonal P-positions *since the previous player to move, i.e., the player who just moved, can force a win. All other positions where the piece is not on the diagonal are called* N-positions *since the next player to move will win by moving the piece to the diagonal. An important property of these positions is that every* N*-position can reach a* P*-position and every* P*-position can only reach* N*-positions.*

This theory is widely applicable to quite a number of games. I recommend the book Winning Ways for Your Mathematical Plays *by Berlekamp, Conway, and Guy for more examples.*

Rough Waters (Medium)

Answer: It turns out that if the exponent of 2 in the prime factorization of n is odd, then the first thinker can force a win, else the second player can win.

Reasoning: Note that if at any point in time a thinker takes neither all nor exactly half of the coins, their opponent can take all of the remaining coins (can you see why?).

We can represent any n in the form $2^k \times M$, where k is some non-negative number (possibly zero!) and M is an odd number. Using our earlier observation, we can claim that the thinkers will take turns taking half the pile until they can't anymore, i.e., when M coins are left. When a thinker faces an odd number of coins, whatever number of coins they take, the other thinker will be able to take the rest of the pile and win. For even k, it will be the first thinker who is left with an odd pile and so will lose, and for odd k the second player will meet the same fate.

Don't forget about the the special case when $n = 2^k$ and $M = 1$. Here, the thinkers will take turns taking half the pile until exactly 1 coin is left. In this case, no one can take this last coin since the previous move involved reducing a pile of 2 coins to a pile of 1 coin. The same logic from before applies here, though: for even k, the first thinker will be left with the singleton pile, and for odd k it is the second player who is stuck.

As an example, if n is $10^6 = 2^6 \times 5^6$, after halving the pile 6 times, the first thinker will be left with a pile of 5^6 coins, and the second thinker can take the entire pile no matter what the first thinker's move is.

Tsunami (Hard)

Proposition 1. *Let $P_n = (x_n, y_n)$. Then*

$$\tfrac{1}{2}x_n + x_{n+1} + x_{n+2} + x_{n+3} + x_{n+4} = -\tfrac{1}{2} \tag{1}$$

for every $n \geq 1$.

Proof. Note that $x_1 = -1$, $x_2 = 0$, $x_3 = 1$, $x_4 = 0$, $x_5 = -1$, so the equation is true for $n = 1$. Now assume the equation holds true up to n. Given the recurrence $x_{n+5} = \frac{1}{2}(x_n + x_{n+1})$, it follows that

$$\tfrac{1}{2}x_{n+1} + x_{n+2} + x_{n+3} + x_{n+4} + x_{n+5} = \tfrac{1}{2}x_{n+1} + \left(-\tfrac{1}{2} - \tfrac{1}{2}x_n - x_{n+1}\right) + \tfrac{1}{2}\left(x_n + x_{n+1}\right) = -\tfrac{1}{2}.$$

□

Suppose $\lim_{n\to\infty} P_n$ exists. Then, taking the limit of both sides of (1), we obtain

$$\lim_{n\to\infty} x_n = -\tfrac{1}{9}.$$

Similarly, $\frac{1}{2}y_n + y_{n+1} + y_{n+2} + y_{n+3} + y_{n+4} = -\frac{1}{2}$ for $n \geq 1$, so

$$\lim_{n\to\infty} y_n = -\tfrac{1}{9}.$$

Hence,

$$\lim_{n\to\infty} P_n = \boxed{\left(-\tfrac{1}{9}, -\tfrac{1}{9}\right).}$$

Remark 3. *There are different ways to show $\lim P_n$ exists; for example, it is sufficient to show that $p(x) = x^4 + x^3 + x^2 + x + \frac{1}{2}$ has 4 distinct roots with modulus less than 1.*

Department of Mathematics, University of Pittsburgh, Pittsburgh, PA 15260
E-mail addresses:
rtp26@pitt.edu
eoh2@pitt.edu
anr216@pitt.edu

Pittsburgh Interdiscip. Math. Rev. is published by the University Library System of the University of Pittsburgh.