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When a set theorist hears "combinatorics": Infinite Ramsey theory

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Abstract

Ramsey theory studies how to find highly-ordered substructures within an otherwise unwieldy object. Ramsey theory is a highly active area of research in contemporary mathematics, with some mathematicians focusing on finite structures and others on infinite ones. In this survey paper, we will give an overview of a few topics in infinite Ramsey theory, with an emphasis on how set theory is involved. That is, we will focus on large, infinite objects and ask exactly how infinite they must be in order to ensure that we have infinite, highly-ordered substructures. After introducing the general idea in the finite case, we will prove Ramsey's theorem about infinite graphs. Then we will transition into questions about finding uncountably infinite, highly-ordered substructures. This will give us a convenient excuse to discuss infinities and independence results in set theory, as well as topological colorings. No knowledge of set theory or topology is required to understand this paper.

Contents

1. **Introduction**

Ramsey theory is a branch of combinatorics and focuses on finding highly ordered (and hence simple) structures within a larger, more complicated structure. For example, suppose we are interested in studying graphs[1] today – and why wouldn't we be? Graphs can be quite messy and complicated, and hence we are interested in finding large subgraphs that are very simple. Two kinds of simple subgraphs are **cliques** (in which any two nodes in the subgraph are connected) or **independent sets** (in which no two nodes in the subgraph are connected).

Take a look at Figure 1 below. Notice in that graph that the nodes A, B, C form a clique, since any two of them are connected (A to B, A to C, and B to C). Also notice that the nodes A, D, E form an independent set, since no two of them are connected (A is not connected to D, A is not connected to E, and D is not connected to E). By contrast, the subgraph with A, C, D is neither a clique nor an independent set. Indeed, it is not a clique since A is not connected to D, and it is not an independent set because A is connected to C.

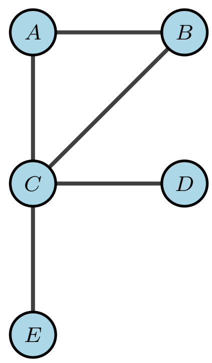

Figure 1: A graph with vertices A through E.

We can summarize what's simple about cliques and independent sets by saying that the edge relation is **uniform** for these subgraphs. Let's drive the point home by illustrating what a clique with four nodes looks like, and then an independent set with four nodes. See Figure 2 for the clique of size four.

Likewise, see Figure 3 for the picture of an independent set with four nodes (no edges in sight!).

I invite the reader to pause here for a moment and to think about how simple it is to describe the above two graphs. For the first, all you have to say is "I have 4 nodes and they are all connected." The second is likewise very simple to describe. And to

[1] We're talking about simple, undirected graphs.

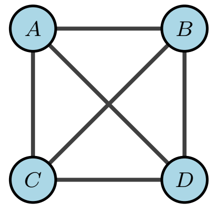

Figure 2: A clique with four nodes

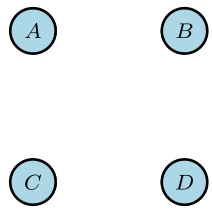

Figure 3: An independent set with four nodes

repeat a point made earlier: these subgraphs are simple *because the edge relation is uniform.* Before continuing, we'll introduce the following standard vocabulary:

Definition 1.1. A subgraph of a graph G is *homogeneous* if it is either a clique or an independent set (with the edge relation inherited from G).

Remark 1.2. *There is another very common way to look at this matter. Some folks prefer to think about edge colorings on a complete graph. That is, start with n nodes and connect them all. Color each* ***edge*** *red or blue. With this language, you'd be interested in finding* monochromatic *subsets. There's no loss of information in writing things as we have so far, or in terms of edge colorings. But it is important that you're aware that there are other ways of describing what's happening.*

Key Theme of Ramsey Theory

This now brings us to the key theme of Ramsey theory: **every large enough graph has a large homogeneous set.** Put differently,

($*$) Every graph with enough vertices must have either a large clique or a large independent set.

Let's illustrate this by using an example with less mathematical language. Suppose you are hosting a raucous party. You'd like to know how many people you must invite in order to ensure that there are three people – let's be imaginative and call them a, b, c – so that either

1. a and b have met before, a and c have met before, and b and c have met before; OR

2. none of a, b, c have ever met each other.

In other words, how many people must you invite in order to guarantee that there must be a homogeneous set of size 3?

It turns out that inviting five people is not enough, since the configuration in Figure 4 is logically possible.

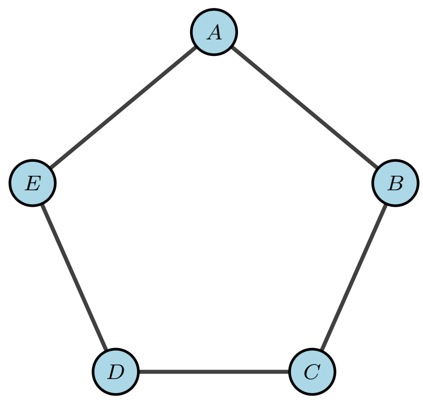

Figure 4: Five people is not enough

In Figure 4, an edge between two nodes means that those two people have met each other. So for example, A and B have met before, but A and C have not met before.

Now why does this graph not have a clique of size 3? Suppose we take *any* three nodes from this graph, and we will show that there are two that are not connected. Since the graph is symmetric, we may suppose for concreteness that A is one of the nodes. Let's look at some cases:

- If B is the next (in alphabetical order) node of the three, then any possible third node is either disconnected from B or from A.

- If C is the next node of the three, then C is not connected to A, and likewise if D is the next node.

You are invited to see that no matter which three nodes you pick, there will be two of them that *are* connected, and hence there are no independent sets of size 3 either.

The following fact shows that 6 is the smallest number needed to ensure that we have a homogeneous set of size 3, *no matter what the graph looks like.* This is known as the **Ramsey Number** $R(3, 3)$.

Fact 1.3. *6 is the smallest number of people you need to invite to ensure that either*

(a) there are three people who have all met, or

(b) there are three people no two of whom have met.

Proof. We start by choosing a person A at random. Consider the other 5 guests. We will split them into two groups: those who know A and those who don't. By the pigeonhole principle, there must be either 3 people who know A or three people who don't.[2] Let's suppose for concreteness that there are three other people who know A, and we call these people X, Y, and Z. See Figure 5 for a picture of the situation so far, where the dotted lines indicate that we don't yet know whether there are edges between the given nodes.

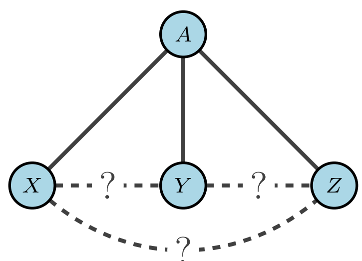

Figure 5: Six guests suffices: First use the Pigeonhole Principle

We then have some subcases: if there are at least two people among X, Y, Z who know each other, then we have a clique of size 3. Suppose for concreteness that X knows Y. Then A, X, Y form a clique, as depicted in Figure 6 below.

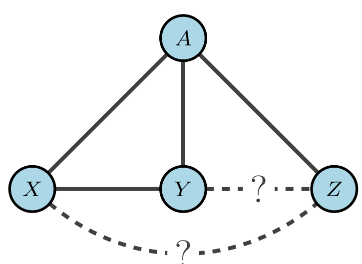

Figure 6: Six guests suffices: Now we have a clique in this case

By contrast, if none of X, Y, or Z know each other, then they form an independent set. In either case, we're done! □

[2]Note that this is where we are using that we have 6 guests: if we only had 5, then we wouldn't be able to push through this stage in the argument!

It turns out that for any k (specifying the size of a homogeneous set), there is a large enough n so that inviting n people to your party *guarantees* that there is either a clique of size k or an independent set of size k. The least such n is known as the **Ramsey number** $R(k,k)$.[3] A few brief comments about the history are in order here. Ramsey proved his famous theorem about infinite graphs and infinite homogeneous sets in his 1930 paper (see [12]). In that paper, he also made some comments about these finite versions. However, the main paper that first developed these finite versions came later, in 1935, and is due to Erdős and Szekeres (see [6]). In terms of exposition, we are working from the finite to the infinite, but the history has things in the other order. Now to state the theorem:

Theorem 1.4. *(Erdős and Szekeres, [6]) Let $k \geq 2$ be a natural number. Then there is an n so that for any graph G with at least n vertices, G has a homogeneous set of size k.*

There is much more that one can say about this topic for finite k, but we wish to transition into talking about infinite graphs.

Section Summary 1.5. After reading this section, you should know what homogeneous sets in the context of graphs are. You should also know the central theme of Ramsey theory: any sufficiently large graph has a large homogeneous set. And finally, you should know that $R(3,3) = 6$.

2. **Infinite graphs**

In this section, we begin our discussion of infinite graphs. We saw in the previous section that for any $k \geq 2$ (specifying the size of a homogeneous set), there is a large enough (finite) n so that *every* graph of size n (or more) has a homogeneous set of size k.

Now consider the following question:

Question 2.1. Does every infinite graph have an infinite homogeneous set?

Or to use the analogy of hosting a party: if you invite infinitely many guests, does that guarantee that either there are infinitely many guests who have all met each other, or infinitely many guests no two of whom have met each other?

It turns out that the answer is yes! This is Ramsey's theorem. We will end up giving the proof in full detail, but first we're going to translate into the language of

[3] It is known that $R(4,4) = 18$ ([10]) and that $R(5,5)$ is between 43 and 49 ([7] and [11]). There are asymmetric versions $R(k,l)$ as well. All of this to say: the discussion for finite k is far from over!

colorings of pairs of natural numbers. Once we complete that translation and give a few examples, we will launch into the proof of Ramsey's theorem.

Definition 2.2. Let X be a nonempty set. We use $[X]^2$ to denote the set of $x \subset X$ so that x has exactly 2 elements.

A *2-coloring* of a set X is a function $c : [X]^2 \to \{0, 1\}$ (c for "coloring").

So a 2-coloring c of a set X assigns a "color" – either 0 or 1 – to each pair of distinct elements from X, but without considering the order of the two elements. You can think of 0 and 1 as red and blue if you'd like.[4]

One can also consider m-colorings instead of just 2-colorings (for $m \geq 2$). We're focusing on 2-colorings since that is enough to introduce and motivate the topic.

Though that is the precise definition of a 2-coloring, it is in practice easier to think of a 2-coloring as follows:

Remark 2.3. *A 2-coloring c of a set X is the "same" as a symmetric function c' defined on $\{(a, b) \in X \times X : a \neq b\}$. That is, an input to c' is a pair of* distinct *elements from X, and if $a, b \in X$ are distinct, then $c'(a, b) = c'(b, a)$.*

We can also go back-and-forth between 2-colorings and graphs. Here's an example: suppose that X is the set $\{A, B, C, D\}$. We'll think of a 2-coloring as a symmetric function as in the previous remark. So suppose we define

$$c(A, B) = c(B, D) = c(A, D) = c(A, C) = 1$$

and define c to be 0 otherwise. We get the same information by considering a graph with A, B, C, and D as vertices and defining two vertices to be connected exactly when c gives 1 for that pair. See Figure 7.

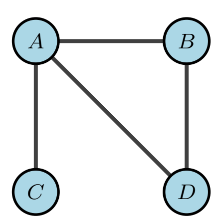

Figure 7: Back-and-forth from colorings to graphs

Continuing, we now define more precisely what we mean by a homogeneous set in the context of colorings:

Definition 2.4. Given a set X and 2-coloring c on X, a subset $H \subseteq X$ is said to be *homogeneous for c* if c is constant on $[H]^2$.

[4]Or, say, mauve and chartreuse – but not the French liqueur.

That is, $H \subseteq X$ is homogeneous for c when there exists $j \in \{0,1\}$ so that for every $x \in [H]^2$, $c(x) = j$. **The order of the quantifiers is crucial!**

Remark 2.5. *Homogeneous sets for 2-colorings correspond to cliques or independent sets in the context of graphs.*

Now we can state Ramsey's theorem more precisely.

Theorem 2.6. *(Ramsey, [12]) Given any 2-coloring $c : [\mathbb{N}]^2 \to \{0,1\}$ of the natural numbers, there exists an infinite $H \subset \mathbb{N}$ so that H is homogeneous for c.*

To make the proof as perspicuous as possible, we'll divide it into two stages. In the first stage, we work to prove the existence of something called a **tail homogeneous set**. This is the poor man's version of a homogeneous set. *Then* we finish the proof by refining the tail homogeneous set into a truly homogeneous set.

Definition 2.7. Let c be a 2-coloring of $\mathbb{N}$, and let $a_0 < a_1 < a_2 < \dots$ be an infinite increasing sequence of natural numbers. We say that a_k is a *friendly* element in the sequence if for all n with $k < n$, $c(a_k, a_n) = 1$. We say that a_k is a *lonely* element in the sequence if for all n with $k < n$, $c(a_k, a_n) = 0$.

The sequence $a_0 < a_1 < a_2 < \dots$ is *tail homogeneous* if every element in the sequence is either friendly or lonely.

So a_k is friendly if it is connected to every *later* element, and a_k is lonely if it is *not* connected to *any* later element.

Remark 2.8. *Constructing the “tail homogeneous” set will require applying the pigeonhole principle infinitely-many times. Turning it into a truly homogeneous set will require an additional application of the pigeonhole principle. Thus we can summarize the proof of Theorem 2.6 by saying:*

(∗) *apply the pigeonhole principle $\infty + 1$-many times.*

Our first lemma says that we can in fact refine a tail-homogeneous set into a homogeneous set:

Lemma 2.9. *Suppose that c is a 2-coloring of $\mathbb{N}$ and that $a_0 < a_1 < a_2 < \dots$ is a tail homogeneous sequence for c. Then there is an infinite $H \subset \{a_k : k \in \mathbb{N}\}$ so that H is homogeneous for c.*

Proof. Let c and $a_0 < a_1 < a_2 < \dots$ be fixed. By definition of a tail homogeneous sequence, for each k, a_k is either friendly or lonely. Since there are infinitely-many distinct elements on the sequence, we may apply the pigeonhole principle to conclude that either there are infinitely-many friendly elements or infinitely-many lonely elements (or both). Suppose for concreteness that there are infinitely-many friendly elements on the sequence. Define $H := \{a_k : a_k \text{ is friendly}\}$, and we claim that H is homogeneous for c with color 1.

This means that we need to show that for any two distinct elements $x, y \in H$, $c(x, y) = 1$. So fix x and y from H with $x \neq y$. Let $x = a_k$ and $y = a_n$. By relabelling if necessary, assume that $k < n$, i.e., that a_n occurs later in the sequence. Now recall that a_k is a friendly element of the original sequence. Thus for any $m > k$, $c(a_k, a_m) = 1$. In particular, since $n > k$, we conclude that $c(a_k, a_n) = 1$. Since $x = a_k$ and $y = a_n$ were arbitrary, we've shown that H is an infinite homogeneous set for c. □

Now we turn to the more difficult and interesting part: constructing a tail homogeneous sequence. This is a recursive construction, but it can be challenging to see at first how it works and why one would want to do the construction in this way. Hence, I'm going to do something here which is not efficient, in terms of space, but which serves a helpful didactic purpose: in the proof, I will begin by writing out the first two steps of the recursion. Then I will start the proof over, making it more formal and precise.

Lemma 2.10. *Suppose that c is a 2-coloring of $\mathbb{N}$. Then there is a tail homogeneous sequence $a_0 < a_1 < a_2 < \dots$ for c.*

Proof. **First Pass**: we'll describe the first couple of steps of the recursion.

We begin by setting $a_0 = 0$. Now observe that for any $k > 0$, either $c(a_0, k) = 0$ or $c(a_0, k) = 1$. Let's split $\{1, 2, 3, 4 \dots\}$ into those k so that $c(a_0, k) = 0$ and those so that $c(a_0, k) = 1$. Apply the pigeonhole principle to see that there are either infinitely-many $k \geq 1$ with $c(a_0, k) = 0$ or there are infinitely-many $k \geq 1$ with $c(a_0, k) = 1$ (or both). Let R_0 be such an infinite set. Here "R" stands for *reservoir.*

Suppose for concreteness that every $k \in R_0$ is connected to $a_0 = 0$, i.e., that $c(a_0, k) = 1$ for all $k \in R_0$. As we continue the construction, we will pick **all** of our future points $a_1 < a_2 < \dots$ from R_0. Every single future point we place on the sequence will come from R_0. This means that every later a_n will be connected to a_0, and hence that a_0 will be a friendly element in the resulting sequence – as long as we keep picking points from R_0. (If we have $c(a_0, k) = 0$ for all $k \in R_0$, then a_0 will turn out to be a lonely element in the sequence.)

At future stages in the proof, we will do two things: we will shrink our infinite reservoir (while keeping it infinite), and we will pick a new point. Here's what this looks like in the next stage: let a_1 be the least element of R_0. Note that $a_0 < a_1$.

Apply the pigeonhole principle again to split $R_0 \setminus \{a_1\}$ (i.e., all those points in R_0 above a_1, since a_1 is the least element) into two sets: Either there are infinitely-many $k \in R_0$ so that $c(a_1, k) = 0$ or there are infinitely-many $k \in R_0$ so that $c(a_1, k) = 1$. Let R_1 be such an infinite set, and suppose for concreteness that $c(a_1, k) = 0$ for all $k \in R_1$. It is crucial to observe that since $R_1 \subset R_0$, when we pick future points from R_1, we are still picking points from R_0, and hence we maintain that all future points are connected to a_0. But we also ensure that no future points are connected to a_1. In other words, a_1 will become a lonely element in the sequence, and a_0 will stay a friendly element in the sequence *as long as we keep picking points from* R_1.

Second Pass We now give the more precise proof. We define by recursion a sequence $(a_0, R_0), (a_1, R_1), (a_2, R_2) \dots$ satisfying the following recursion hypotheses (which have cute names) for each $m \in \mathbb{N}$:

1. for all $k < m$, $a_k < a_m$ (points increasing);
2. $R_m \subset \mathbb{N}$ is infinite and $a_m < \min(R_m)$ (plentiful reservoir);
3. for all $k < m$, $R_m \subset R_k$ (shrink reservoir);
4. either
   - (4a) for all $n \in R_m$, $c(a_m, n) = 0$ (a_m eventually lonely) OR
   - (4b) for all $n \in R_m$, $c(a_m, n) = 1$ (a_m eventually friendly).
5. If $m \geq 1$, then $a_m \in R_{m-1}$.

To start the recursion, we define $a_0 = 0$ and apply the pigeonhole principle to find an infinite $R_0 \subset \{1, 2, 3, 4, \dots\}$ so that either (4a) or (4b) above is true. Note that items (1), (3), and (5) are vacuous in the present case, since $m = 0$. This defines (a_0, R_0) satisfying the above recursion hypotheses.

Now suppose that we've defined $(a_0, R_0), \dots, (a_m, R_m)$, and we define the next pair (a_{m+1}, R_{m+1}). Let $a_{m+1} = \min(R_m)$. By condition (2) applied to m, this secures condition (1) for $m+1$. Apply the pigeonhole principle to the infinite set $R_m \setminus \{a_{m+1}\}$ to define an infinite $R_{m+1} \subset R_m$ so that either $c(a_{m+1}, n) = 0$ for all $n \in R_{m+1}$ or so that $c(a_{m+1}, n) = 1$ for all $n \in R_{m+1}$. This secures the remaining conditions for $m + 1$. Since m was arbitrary, this completes the construction of the sequence.

We now show that $a_0 < a_1 < a_2 < \dots$ forms a tail homogeneous sequence. Fix $k \in \mathbb{N}$, and we will show that either a_k is lonely in the sequence or that a_k is friendly in the sequence. Consider the reservoir R_k associated to a_k. By our recursion hypotheses, we know that either for all $n \in R_k$, $c(a_k, n) = 0$ or that for all $n \in R_k$, $c(a_k, n) = 1$. Let's suppose for concreteness that for all $n \in R_k$, $c(a_k, n) = 0$. We claim that a_k is lonely in the sequence. Thus fix an arbitrary larger $n > k$, and we show that $c(a_k, a_n) = 0$. By recursion hypothesis (5), we know that $a_n \in R_{n-1}$. By recursion hypothesis (3), we also know that $R_{n-1} \subseteq R_k$. Thus $a_n \in R_k$. Therefore, $c(a_k, a_n) = 0$, as we intended to show. □

We can now finish the proof of Theorem 2.6 by stitching the above two lemmas together:

Proof. (Of Theorem 2.6) Let c be an arbitrary 2-coloring on $\mathbb{N}$. By Lemma 2.10, we may construct a sequence $a_0 < a_1 < a_2 < \dots$ which is tail homogeneous for c. By applying Lemma 2.10, we can find an infinite $H \subset \{a_n : n \in \mathbb{N}\}$ so that H is homogeneous for c. □

Remark 2.11. *Ramsey's middle name was Plumpton. Poor guy.*

It's worth drawing out a couple of corollaries from this, the first of which is a familiar result from an intro to analysis course:

Corollary 2.12. *Let $(a_n : n \in \mathbb{N})$ be a sequence of real numbers. Then it has a monotonic subsequence.*

Proof. Define a coloring c on $\mathbb{N}$ by fixing arbitrary $k < n$ and setting $c(k, n) = 0$ iff $a_k > a_n$ and setting $c(k, n) = 1$ otherwise. Let $H \subset \mathbb{N}$ be an infinite homogeneous set for c, as guaranteed by Theorem 2.6. Suppose that c takes the constant value 1 on $[H]^2$. Then for any $k < n$ in H, we have $c(k, n) = 1$ and hence $a_k \leq a_n$. This implies that the sequence $(a_n : n \in H)$, where we restrict to those elements whose indices are in H, is monotonic. □

Another corollary:

Corollary 2.13. *Let G be a graph with infinitely-many vertices. Then G has an infinite homogeneous set.*

Proof. Let $v_0, v_1, v_2, \dots$ list out infinitely-many pairwise distinct vertices from G. Define a coloring c on $\mathbb{N}$ by fixing arbitrary $k \neq n$ and setting $c(k, n) = 0$ iff v_k and v_n are not connected in the graph. Let $H \subset \mathbb{N}$ be an infinite homogeneous set for c. Arguing as in the previous corollary, we see that $\{v_n : n \in H\}$ is the desired homogeneous set (a clique if c takes constant value 1 on $[H]^2$ and an independent set if c takes constant value 0 on $[H]^2$). □

Section Summary 2.14. After reading this section, you should know how to go back-and-forth from graphs to 2-colorings. Then you should know that Ramsey's theorem shows that every infinite graph has an infinite homogeneous set, and you should know that the proof involves “∞+1”-many applications of the pigeonhole principle. If you are particularly interested in the topic, you should at least know that the proof of Ramsey's theorem given here involves two stages: first constructing a tail homogeneous set and then thinning it to a homogeneous set.

3. Sadness and newfound joys: Uncountable infinities

This is the point where we start to introduce some set theory. We'll first make a couple of remarks about infinite sizes in general, before developing the theme of finding larger and larger homogeneous sets in a graph. We begin by recalling when two sets have the same cardinality (i.e., size):

Definition 3.1. Two sets X and Y have the *same cardinality* iff there is a bijection $f : X \to Y$.

A bijection acts as a one-to-one correspondence between the elements of X and those of Y.

Definition 3.2. We say that a set X is *countably infinite* if X has the same size as $\mathbb{N}$. If X is infinite but not countably infinite, then we say that X is *uncountably infinite*, or more simply *uncountable*.

What's perhaps surprising upon first making the aquaintence of these ideas is that two infinite sets X and Y can have the same size, even if in some sense there are “more” things in Y than in X. Here are some examples:

Example 3.3. $\mathbb{N} \times \mathbb{N}$, $\mathbb{Q}$, and $\mathbb{Z}$ are all countably infinite.

One might then think that *every* infinite set is countable, but this is not the case. In 1891, Georg Cantor proved that the set of real numbers $\mathbb{R}$ has a strictly larger infinite size than that of the natural numbers:

Theorem 3.4. *(Cantor, [3])* $\mathbb{R}$ *is uncountably infinite. That is to say, there is no bijection from* $\mathbb{N}$ *to* $\mathbb{R}$*.*

Before we refine our question, recall that the key idea of Ramsey theory is that any large enough graph has a large homogeneous set. Moreover, we know from Theorem 2.6 that any infinite graph has an infinite homogeneous set.

Question 3.5. Does every uncountably infinite graph have an uncountably infinite homogeneous set? Put differently, if X is uncountably infinite and $c : [X]^2 \to \{0, 1\}$ a 2-coloring, does c have an uncountably infinite homogeneous set?

And now the following lachrymose result:

Theorem 3.6. *(Sierpinski, [13]) There is a coloring $c : [\mathbb{R}]^2 \to \{0, 1\}$ so that c has no uncountably infinite homogeneous set.*

We're going to prove a slightly weaker version of this theorem, where we come up with a coloring defined on some uncountable $X \subset \mathbb{R}$; this turns out to actually not be too difficult. But before proving the restricted version of the theorem, we need to develop a bit of the theory of **cardinal numbers**.

Discussing cardinal numbers will help us refine our questions about how large of a homogeneous set we're looking for and about how large of a graph we need before having a homogeneous set of such a size. Moreover, the proof of Sierpinski's theorem will require us to have a slightly more refined understanding of cardinalities.

A cardinal number is a size number. The first ones are: 0, 1, 2, 3, ... and so on. That is to say, the natural numbers (including 0) are all cardinal numbers. This makes sense: when we count finite numbers of things, we use the natural numbers to do so. After these finite sizes comes the first infinite cardinal number. This is denoted $\aleph_0$, and it is the size of $\mathbb{N}$.

After this, there is a next infinite size, $\aleph_1$. Then a next one, $\aleph_2$, followed by $\aleph_3$, $\aleph_4$, and so on. You can then take a limit of that sequence, call the limit $\aleph_\omega$, and start the process over: $\aleph_{\omega+1}$, $\aleph_{\omega+2}$, and so on. A key point is that for any cardinal number, there is always a next highest cardinal number. This "$\aleph$-sequence" acts as a ruler for cardinalities of all infinite sets: ZFC set theory proves that for any infinite set X, there is some cardinal number $\aleph_\alpha$ so that X has the same size as $\aleph_\alpha$.

We will be focusing on $\aleph_1$ in our discussion of Theorem 3.6. However, we're going to do an annoying thing that set theorists do and **write ω_1 rather than $\aleph_1$.** This is more for psychological purposes: writing ω_1 emphasizes to many people the order structure on the set.

What you need to know about ω_1 in this paper can be summarized in the following:

Fact 3.7. *The following are standard facts about ω_1.*

1. *ω_1 is an uncountably infinite set.*
2. *ω_1 is the least size of an uncountable set. Hence, if X is uncountable, there is an injection $f : \omega_1 \to X$.*

3. *There is an order on ω_1 (in fact, just the membership relation $\in$), which we denote by $\lhd$, so that $(\omega_1, \lhd)$ is a well-order.*[5]

4. *Hence, if $\alpha \in \omega_1$, then there is a successor element to α in the $\lhd$ ordering; we write $\alpha + 1$ for this element.*

Now we can begin the proof of the weakening of Theorem 3.6:

Proof. (of a restricted version of Theorem 3.6) We first fix an injection $f : \omega_1 \to \mathbb{R}$ as guaranteed by Fact 3.7 (2). Let $r_\alpha = f(\alpha)$ for each $\alpha \in \omega_1$. We will define a 2-coloring c on $\{r_\alpha : \alpha \in \omega_1\}$ (i.e., on the range of f) by comparing whether or not the ordering on ω_1 agrees with the ordering on $\mathbb{R}$. To put it another way, color 1 will mean that f *preserves* the order, and color 0 will mean that f *reverses* the order.

Consider, then, $\alpha, \beta \in \omega_1$ with $\alpha \lhd \beta$. So α is less than β in the ordering on ω_1. Since f is an injection, $r_\alpha = f(\alpha) \neq f(\beta) = r_\beta$. Recalling that r_α and r_β are real numbers, we conclude that either $r_\alpha < r_\beta$ or $r_\alpha > r_\beta$. We will color $\{r_\alpha, r_\beta\}$ with 1 if $r_\alpha < r_\beta$ (so that both $\alpha \lhd \beta$ and $r_\alpha < r_\beta$) and color 0 otherwise.

Let's look at some pictures. Figure 8 shows the case when f is order-preserving, and hence we have color 1. By contrast, Figure 9 shows the case when f is order-reversing and c gives color 0:

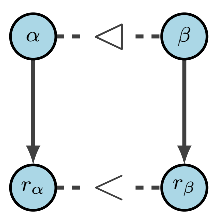

Figure 8: The picture when f is order-preserving

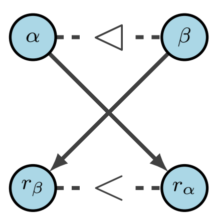

Figure 9: The picture when f is order-reversing

This defines our coloring and hopefully gives some intuition about what is happening. We now show that no uncountable subset of $\{r_\alpha : \alpha < \omega_1\}$ is homogeneous. Here

[5]Recall that a well-order is a linear order in which every non-empty set has a least element. The natural numbers are an example of a well-order; the integers with the usual $<$ are *not* a well-order.

it is a bit easier to think about the indices: suppose we have $H \subset \omega_1$ uncountable so that $\{r_\alpha : \alpha \in H\}$ is homogeneous for the coloring c. Consider first the case when c takes constant value 1. We claim that the function taking $\alpha \in H$ to r_α is order preserving. Indeed, if $\alpha \lhd \beta$ are in H, then $c(r_\alpha, r_\beta) = 1$, and hence $r_\alpha < r_\beta$.

Now we can get our contradiction. For each $\alpha \in H$, let $\text{next}(\alpha)$ denote the next element of H above α; this is one of the places where we're using that $\lhd$ is a well-order on ω_1. Note that if $\alpha \lhd \beta$ are both in H, then $\text{next}(\alpha) \unlhd \beta$. For each $\alpha \in H$, since $r_\alpha < r_{\text{next}(\alpha)}$ are real numbers, we can apply the density of the rationals to choose a rational number q_α so that $r_\alpha < q_\alpha < r_{\text{next}(\alpha)}$. See Figure 10 for the picture.

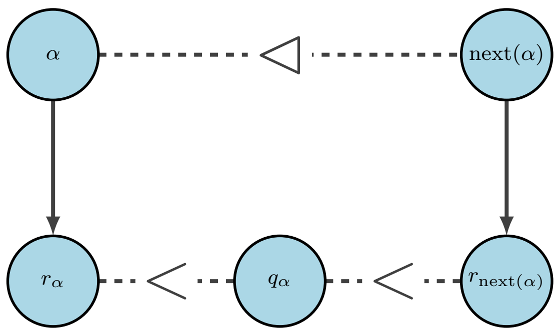

Figure 10: Pick a rational q_α between r_α and $r_{\text{next}(\alpha)}$

We now claim that the function taking $\alpha \in H$ to q_α (remember that q_α is rational) is an injection; note that since H is uncountable this contradicts that $\mathbb{Q}$ is countable! Thus fix $\alpha \lhd \beta$ from H. Then we have the following: $r_\alpha < q_\alpha < r_{\text{next}(\alpha)}$ and $r_\beta < q_\beta < r_{\text{next}(\beta)}$. But $\text{next}(\alpha) \unlhd \beta$, and hence $r_{\text{next}(\alpha)} \leq r_\beta$. Therefore we have

$$q_\alpha < r_{\text{next}(\alpha)} \leq r_\beta < q_\beta,$$

and therefore $q_\alpha < q_\beta$.

This almost finishes the proof. The other case is that the coloring takes the constant value 0. The only difference of difference in this case is that the ordering of the real numbers r_α gets reversed. □

Let's now pause and review where we are in the story. We'd like to prove theorems that say: if you have a graph that has enough nodes, then you get a large homogeneous set. We've seen the following:

1. for each finite $k \geq 3$, there is a finite number n so that any graph with at least n nodes has a k-sized homogeneous set (Theorem 1.4);

2. every infinite graph has an infinite homogeneous set (Theorem 2.6);

3. it is not true that every *uncountably* infinite graph has an *uncountably* infinite homogeneous set (Theorem 3.6).

One of a large number of remarkable results in this field is that there actually is a large enough cardinal number κ so that any graph with at least κ-many nodes has an ω_1-sized homogeneous set. This result is due to Erdős and Rado.

Before we state this theorem, we need to go over a bit more set theory. You recall from Theorem 3.4 that the real numbers $\mathbb{R}$ are uncountable. That is, there is no bijection from $\mathbb{N}$ to $\mathbb{R}$. It turns out that $\mathbb{R}$ is in bijection with the *powerset* of $\mathbb{N}$, which we denote by $\mathcal{P}(\mathbb{N})$:

Fact 3.8. *There is a bijection from* $\mathbb{R}$ *to* $\mathcal{P}(\mathbb{N})$*.*

Next, we want to see that $\mathcal{P}(\mathbb{N})$ is in bijection with all countable binary sequences, i.e., all $f : \mathbb{N} \to \{0, 1\}$. The bijection associates a $A \subseteq \mathbb{N}$ with its characteristic function $\chi_A : \mathbb{N} \to \{0, 1\}$ defined by $\chi_A(n) = 1$ iff $n \in A$.

The point of all this is to explain the following: we will use $2^{\aleph_0}$ to denote the size of the powerset of $\mathbb{N}$. This is therefore the size of $\mathbb{R}$ as well:

Fact 3.9. $2^{\aleph_0}$ *is the cardinality of* $\mathbb{R}$*.*

We're almost ready to state the Erdős-Rado theorem. The final bit of notation we need is the following: given a cardinal number κ, we use κ^+ to denote the least cardinal bigger than κ. Earlier when we were discussing the $\aleph$-sequence, we had $\aleph_0$, then $\aleph_1$, $\aleph_2$, etc. So we can use this "plus" notation to say, for example, that $\aleph_0^+ = \aleph_1$, and so forth. Putting all of this together, we will write $(2^{\aleph_0})^+$ to denote the next largest cardinal number above $2^{\aleph_0}$. Let's state the theorem:

Theorem 3.10. *(Erdős-Rado, 1956, [5]) Suppose that* G *is a graph with at least* $(2^{\aleph_0})^+$*-many nodes. Then* G *has either a clique of size* ω_1 *or an independent set of size* ω_1*.*

Their theorem is more general, but this is enough for us to appreciate how interesting this is. Remember that, without additional assumptions on the graph, $(2^{\aleph_0})^+$ is optimal, since there are graphs of size $2^{\aleph_0}$ which do not have uncountable homogeneous sets (Theorem 3.6).

Section Summary 3.11. After reading this section, you should know that it is not true that every uncountable graph/coloring has an uncountable homogeneous set. However, if you consider graphs of size $(2^{\aleph_0})^+$ (the successor of the size of $\mathbb{R}$), then you are guaranteed an uncountable homogeneous set.

4. **From joy to befuddlement: Independence results**

But not too fast. Having stated the Erdős-Rado theorem, we have now run into the territory of independence results in set theory. In this section, we're going to explain what is meant by "independence results" and discuss the Continuum Hypothesis. We will then use these ideas to highlight some of the odd features of the Erdős-Rado theorem.

Many of the math folk I've talked to often get hung up on the notion of "independence". However, it is something that we're all already intimately familiar with, though perhaps not by that specific term. Let's ease into the idea by considering some examples.

Example 4.1. When you take an intro to abstract algebra course, you spend some time defining what a group is. You then study different kinds of groups, naturally enough. We distinguish, say, between Abelian and non-Abelian groups, the former being commutative (like addition in the integers) and the latter failing to be commutative (like multiplication of invertible matrices).

Consider, then, the statement "the group operation is commutative", or in symbols "$\forall x, y\ (x \cdot y = y \cdot x)$". Let's denote this by φ_{Ab} ("Ab" for "Abelian"). φ_{Ab} is *independent* of the axioms of group theory.

All that this means is that the basic, starting axioms of a group do not prove that φ_{Ab} is true and also do not prove that φ_{Ab} is false. Why? Because there are groups where φ_{Ab} is true (namely, Abelian groups) and ones where it is false.

Definition 4.2. A statement φ is *independent* of a set A of axioms if A does not prove that φ is true and if A does not prove that φ is false.

Here's how this relates to set theory and Ramsey theory. In the previous section, we introduced the symbol $2^{\aleph_0}$, noting that it is the size of $\mathbb{R}$. Since $\mathbb{R}$ is uncountable, we are justified in writing $\aleph_0 < 2^{\aleph_0}$. Notice, however, that this inequality does not tell us *how much* bigger $2^{\aleph_0}$ is than $\aleph_0$. For instance, is it the very next size, $\aleph_1$? Or is it $\aleph_2$? How large is the gap between $\aleph_0$ and $2^{\aleph_0}$? Put differently, can we find some $X \subseteq \mathbb{R}$ so that

(a) X is uncountable but

(b) X has a smaller size than $\mathbb{R}$?

We're therefore asking the question: are there any sizes in between $\aleph_0$ (countable infinity) and $2^{\aleph_0}$ (the size of $\mathbb{R}$)?

It turns out that this question cannot be decided just based upon the standard axioms of set theory, namely the ZFC axiom system. Let's make this a bit more precise:

Definition 4.3. The *Continuum Hypothesis*, denoted CH, states that $2^{\aleph_0} = \aleph_1$.

Thus the CH asserts that $2^{\aleph_0}$, the size of $\mathbb{R}$, is the smallest possible size it could possibly be, namely $\aleph_1$. To say that the CH is independent of ZFC is to say both that ZFC does not prove the CH and that it does not refute the CH. We begin with the latter of these, a result due to Gödel.

Theorem 4.4. *(Gödel, [9]) The* CH *is consistent with* ZFC*.*

Gödel proved this theorem by building a model L, now known as Gödel's **constructible universe**, in which the axioms of ZFC are true and in which the CH is also true. L is constructed recursively, level-by-level. At each stage in the construction, you only add the minimum possible amount of information, and this ensures that at the end of the construction, you have only added what *logically must exist* in a model of set theory.[6] In other words, L contains only the bare minimum guaranteed by the ZFC axioms, and hence it is the "smallest" model of set theory. A closer analysis of L then shows that L only has $\aleph_1$-many real numbers,[7] and hence L satisfies the CH!

The other part of the independence result came later and is due to Cohen:

Theorem 4.5. *(Cohen, [4])* ZFC *does not prove that the* CH *is true. That is, there is a model*[8] *of* ZFC *in which $2^{\aleph_0} = \aleph_2$, and hence, in that model, there are infinite sizes in between $\aleph_0$ (countable infinity) and $2^{\aleph_0}$ (the size of $\mathbb{R}$).*

In fact, using forcing, you can create a model in which $2^{\aleph_0}$ is any $\aleph_\alpha$ as long as α does not have "countable cofinality". Whatever this precisely means, we can make models in which $2^{\aleph_0}$ is $\aleph_3$ or $\aleph_{42}$ or $\aleph_{\omega+42}$ etc.

We'll make a few comments about forcing, to give you a feel for the idea, before connecting these ideas with the Erdős-Rado theorem. Cohen invented the technique of forcing in order to prove the above theorem. Forcing involves starting with a model of ZFC and then adding additional, new elements to the model. By adding these new elements and then closing under the usual set-theoretic operations (like pairing, union, etc.), we obtain a larger model of ZFC. However, while we preserve

[6]We're talking transitive, well-founded models here.

[7]More precisely, L has a bijection from its set of real numbers to the thing that it thinks is $\aleph_1$.

[8]We're a bit imprecise here. It's a *relative* consistency result that Cohen proved, namely: IF ZFC is consistent, then so is ZFC with the negation of the CH. Mutatis mutandis for Gödel's result.

the base theory of ZFC in the larger model, in the process of extending the starting model, we might change the truth values of other statements – like the CH!

As an analogy, consider the case of field extensions. In the field $\mathbb{Q}$ of rational numbers, we don't have a solution to the polynomial equation $x^2 - 2 = 0$. That is to say, the square root of 2 is not a rational number. However, we can build a larger field, namely $\mathbb{Q}(\sqrt{2})$ (read "$\mathbb{Q}$ adjoin root two"), which *does* have a solution to $x^2 - 2 = 0$. The elements of $\mathbb{Q}(\sqrt{2})$ all look like $q + \sqrt{2}r$, where q and r are both rational numbers. Note that $\mathbb{Q}$ and $\mathbb{Q}(\sqrt{2})$ are both fields, so that the base theory of "fields" is preserved when extending. However, the statement "$\exists x\ (x^2 - 2 = 0)$" is false in $\mathbb{Q}$ and true in $\mathbb{Q}(\sqrt{2})$.

This analogy therefore illustrates how, in mathematics, we can extend a given object while preserving what kind of thing it is, but while also changing the truth values of other statements of interest. Forcing does this in the context of models of ZFC.

To return to the Erdős-Rado theorem, we know that any graph with at least $(2^{\aleph_0})^+$-many nodes has an ω_1-sized homogeneous set. But without the context of a specific model of ZFC in which we have nailed down exactly what the value of $2^{\aleph_0}$ is, this theorem doesn't answer questions like the following: does any graph with at least $\aleph_2$-many nodes have an ω_1-sized homogeneous set? What about with $\aleph_{42}$-many nodes?

For example, both of the following are consistent with ZFC:

1. It is consistent with ZFC that any graph with at least $\aleph_2$-many nodes has a homogeneous set of size ω_1. This occurs in a model of the CH since then $2^{\aleph_0} = \aleph_1$ and hence $(2^{\aleph_0})^+ = (\aleph_1)^+ = \aleph_2$. Then apply Erdős-Rado.

2. It is consistent with ZFC that any graph with at least $\aleph_{42}$-many nodes has a homogeneous set of size ω_1, but not every graph of size $\aleph_{41}$-many nodes has a homogeneous set of size ω_1. This occurs in a model in which $2^{\aleph_0} = \aleph_{41}$, since then $(2^{\aleph_0})^+ = (\aleph_{41})^+ = \aleph_{42}$, and we can apply Erdős-Rado. However, because $2^{\aleph_0} = \aleph_{41}$, we can create graphs in that model of size $\aleph_{41}$ which don't have ω_1-sized homogeneous sets (Theorem 3.6).

Section Summary 4.6. After reading this section, you should know what the Continuum Hypothesis says, what it means for a statement to be independent of a set of axioms, and that the CH is independent of the axioms of ZFC set theory. You should also know that applications of the Erdős-Rado theorem are sensitive to the value of $2^{\aleph_0}$ in the model that you're working in.

5. From befuddlement to further progress: Topological colorings

So far the theme has been that by having a large enough graph (or by 2-coloring a large enough set), we can get a large homogeneous set. The emphasis has been on *how many* nodes we need in our graph in order to ensure a predetermined size for a homogeneous set.

In this section, we will consider a different line of attack by coloring objects with some additional structure. We're therefore no longer just worried about cardinality, but also about graphs (or 2-colorings) with additional features.

We will concern ourselves with so-called "topological colorings." As the name suggests, topological features play a role here. You don't need to know any topology to follow this; you only have to be able to draw circles![9]

Suppose that we have an **injective** function $f : \mathbb{R} \to \mathbb{R}$. Using Ramsey's theorem (Theorem 2.6), we can build an infinite subfunction $\bar{f} \subset f$ so that either

1. $\bar{f}$ is strictly increasing, OR
2. $\bar{f}$ is strictly decreasing.

How? One way is to begin by defining the sequence $(a_n : n \in \mathbb{N})$ by $a_n := f(n)$. We then apply Corollary 2.12 to generate a monotonic subsequence $(a_{n_k} : k \in \mathbb{N})$. Suppose for concreteness that the sequence is monotonically decreasing. Then $k < l$ implies $a_{n_k} \geq a_{n_l}$. But $a_{n_k} = f(n_k) \neq f(n_l) = a_{n_l}$, where the equalities hold by definition of the a-points and the inequality since f is injective and since $n_k \neq n_l$. Thus the sequence $(a_{n_k} : k \in \mathbb{N})$ is in fact strictly decreasing.

Suppose, however, that we want to find an *uncountable* $g \subset f$ so that either

1. g is strictly increasing, OR
2. g is strictly decreasing.

To achieve this, it seems that we'll need to be able to get an uncountable homogeneous set for a coloring. But didn't we already see that having even $2^{\aleph_0}$ nodes is not enough to get an uncountable homogeneous set? We did indeed. But that only shows that *some* 2-coloring on $\mathbb{R}$ doesn't work not that *every* 2-coloring doesn't work. So we ask the question: is there something different about the case of building an uncountable monotonic subfunction of an injective function $f : \mathbb{R} \to \mathbb{R}$? And yes, there is! That's what we turn to now.

Fix $f : \mathbb{R} \to \mathbb{R}$ which is injective, and as in, say, calculus, imagine the graph of f (picture coming in a moment). I want to clearly get across the sense in which we have

[9] I hope this does make you want to learn some topology...

a "topological" coloring. What you need to know about topology in the x, y plane is that the most basic objects are going to be open discs whose centers are pairs of rational numbers and whose radii are rational numbers.

(∗) One of the reasons this is significant is that the basic topological objects we're using are pretty simple, and there are only countably-many of them (since $\mathbb{Q}$ is countable).

We will call these **basic open sets**.

What does this have to do with f? Let's suppose that $r < s$ and, for concreteness, that $f(r) < f(s)$ as well (remember that $f(r) \neq f(s)$, so $f(r) > f(s)$ is the other option). Now draw basic open sets around $(r, f(f))$ and $(s, f(s))$ so that all of the y-coordinates of the first disc are well below all of the y-coordinates of the 2nd disc; see Figure 11. Note in Figure 11 that on the y-axis, we have highlighted the projections of these open discs so that you can see that the points in the lower disc are all well below the points in the upper disc.

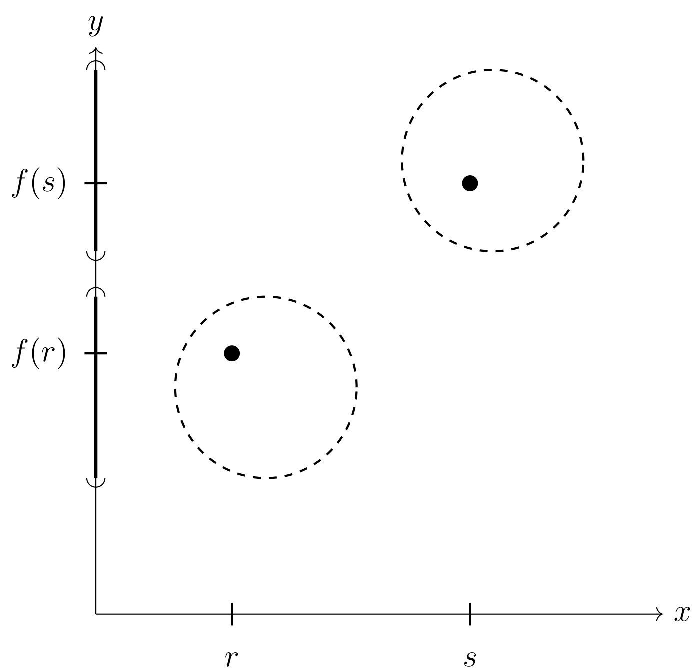

Figure 11: Picking good open sets

Now here's the fun part: suppose we take points $(r', f(r'))$ and $(s', f(s'))$ with $(r', f(r'))$ coming from the lower disc and $(s', f(s'))$ coming from the upper disc, as in Figure 12.

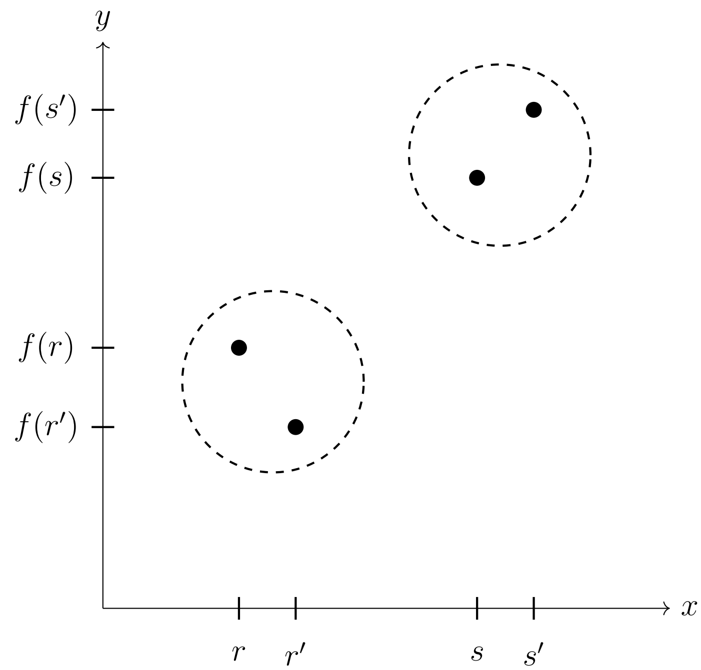

Figure 12: Witnessing the value of the coloring by basic open sets

The key observation is that because of how the discs are arranged, f is order preserving on $\{r', s'\}$! In fact, f is order preserving on each of the following sets: $\{r, s\}$, $\{r, s'\}$, $\{r', s\}$, and $\{r', s'\}$. For example, f is order preserving on $\{r, s'\}$ because $r < s'$ and $f(r) < f(s')$, and f is order preserving on $\{r', s\}$ because $r' < s$ and $f(r') < f(s)$.

The upshot is that the value of the coloring (recall color 1 means order preserving and color 0 means order reversing) on two points $(a, f(a)), (b, f(b))$ from the graph of f is entirely determined by how certain open discs are arranged. In this sense, this coloring is a topological coloring. The additional structure that we have to work with consists of these countably many "rational open discs" (i.e., open discs with rational radius and rational center).

Now let's ask the following:

Question 5.1. Is is true that for every injective $f : \mathbb{R} \to \mathbb{R}$, there is an uncountable monotonic $\bar{f} \subset f$?

However, here we again run into deep questions about independence from ZFC. The statement "every injective $f : \mathbb{R} \to \mathbb{R}$, has an uncountable monotonic $\bar{f} \subset f$" is in fact independent of ZFC. First, we can see that its *negation* is consistent with ZFC by the following theorem and corollary:

Theorem 5.2. *(Sierpinski, [14]) The* CH *implies that there is an* $f : \mathbb{R} \to \mathbb{R}$ *which is not continuous on any uncountable* $A \subseteq \mathbb{R}$.

Corollary 5.3. *The* CH *implies that there is an* $f : \mathbb{R} \to \mathbb{R}$ *which is not monotonic on any uncountable* $A \subseteq \mathbb{R}$*, and hence that the answer to Question 5.1 is "No".*

Proof. Let f be as in Theorem 5.2. It is a standard fact from analysis that any monotonic function $g : B \to \mathbb{R}$, with $B \subseteq \mathbb{R}$, has at most a countable infinity of points of discontinuity.

So suppose for a contradiction that f is monotonic on some uncountable $A \subseteq \mathbb{R}$. Let $A_0 \subseteq A$ be the countable set of points where f is discontinuous on A. Then $A \setminus A_0$ is still uncountable, and f is continuous on $A \setminus A_0$. But this contradicts the assumption that f is not continuous on any uncountable subset of $\mathbb{R}$. □

The upshot of this is that the statement under consideration is *not* provable in ZFC, since it's negation is consistent with ZFC.

Now we briefly comment on the other direction of the independence result. It turns out that this statement is consistent with ZFC, and Abraham, Rubin, and Shelah proved this in the mid-80's:[10]

Theorem 5.4. *(Abraham, Rubin, and Shelah, [2]) It is consistent with* ZFC *that every injective* $f : \mathbb{R} \to \mathbb{R}$ *has an uncountable monotonic* $\bar{f} \subset f$.

They build their model using the technique of "iterated forcing," a detailed description of which is beyond the scope of this article. The rough idea is that they list off all of the possible injective $f : \mathbb{R} \to \mathbb{R}$ and one-by-one, they use forcing to build a larger model in which each has an uncountable, monotonic subfunction. By necessity (Corollary 5.3), their model satisfies that the CH is **false**. In fact, in their model, the value of the continuum is exactly $\aleph_2$, i.e., their model satisfies that $2^{\aleph_0} = \aleph_2$. It remained an open problem for some time whether or not you could build a similar model in which the continuum is even larger, say in which $2^{\aleph_0} = \aleph_3$. Itay Neeman and I solved this in the affirmative several years back:

Theorem 5.5. *(Gilton, Neeman [8]) It is consistent with* ZFC *and with* $2^{\aleph_0} = \aleph_3$ *that every injective* $f : \mathbb{R} \to \mathbb{R}$ *has an uncountable monotonic* $\bar{f} \subset f$.

This is a good point to stop the main thread and summarize what we've covered. The goal of this article was to describe a few of the main themes in infinite Ramsey

[10]It was originally just Abraham and Shelah who proved this in [1]. The three-authored paper proves a slightly better result, and this is the one that most people who are interested in this material know about.

theory. We saw the general idea of a "homogeneous set" in the context of graphs, or equivalently, colorings. We then looked at Ramsey's theorem, which guarantees that we can find infinite homogeneous sets, but we saw that it is a very delicate matter to achieve uncountable homogeneous sets. If we have large enough graphs (namely, of size $(2^{\aleph_0})^+$), then we can guarantee the existence of uncountable homogeneous sets, but no smaller cardinality will guarantee this. Nevertheless, once we impose additional constraints, such as topological ones, on the colorings, we can obtain Ramsey-like theorems which are consistent with ZFC, though they contradict the CH.

Of course, there is plenty more to say, both about the finite and infinite versions. (For one among many examples in the latter category, there is another kind of topological coloring axiom due to Todorčević, [15].) Nevertheless, I hope that this is enough to whet your appetite for Ramsey theory and set theory.

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