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When equivariant homotopy theory meets combinatorics

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Introduction

In many different ways, mathematics often amounts to finding and studying suitable *algebraic structures* on various collections of objects. In any first class in algebra, one gets to know the bestiary of monoids, (abelian) groups, rings, modules, algebras, etc. Algebraic structures are omnipresent and can also be viewed in a broader sense. For instance, certain categories can be endowed with a commutative multiplicative structure, making them into a *symmetric monoidal category*. The latter is a triple $(C, \otimes, \mathbb{1}_C)$, where C is a category, $\otimes$ is a functor $C \times C \to C$, viewed as a multiplication operation, and $\mathbb{1}_C \in C$ is an object in C, representing a unit for the multiplication. We also require several axioms to be satisfied, for example associativity and commutativity of $\otimes$, and unitality of $\mathbb{1}_C$ with respect to $\otimes$. A concrete example is the category of vector spaces over a field k, where $\otimes$ is defined as the tensor product of vector spaces and $\mathbb{1}_C$ as the field k itself.

Viewing $\mathbb{1}_C$ as a functor from the one-point category to C (picking the object $\mathbb{1}_C \in C$), both $\mathbb{1}_C$ and $\otimes$ are functors $C^n \to C$ for $n = 0$ and $n = 2$ respectively. We want to think of such objects as *operations* taking n inputs and producing one output. Our example illustrates that algebraic structures can usually be defined using a set of *operations*, which should satisfy several axioms, also called *relations*. To neatly encode the data of an abstract algebraic structure, viewed as a collection of operations and relations, topologists invented devices called *operads*. Then, a particular instance of this abstract structure, a realization of this operad, i.e. the data of an object endowed with such operations that satisfy the required axioms, is called an *algebra* over this operad. In our example, the abstract algebraic structure in question is that of an associative, commutative, and unital multiplication, and a particular instance is the data of $(C, \otimes, \mathbb{1}_C)$.

Pictorially, an operation with n inputs can be thought of as a tree with n "input branches" and one "output branch". Operations can be composed by plugging the result of an operation as an input for another operation. This corresponds to tree grafting. Also, permutation of the inputs yields an action of the symmetric group S_n on operations in arity n, that is, with n inputs, which corresponds to twisting the tree's branches. Relations between operations can then be expressed in terms of equalities of certain twisted and grafted trees. We will define operads more precisely in Subsection 1.1, and make the connection with this botanical point of view.

As usual, topologists do not want to distinguish between homotopy equivalent objects. Thus, one is led to consider algebraic structures where the relations between the operations are satisfied not *on the nose*, but rather *up to homotopy*. More precisely, writing our relations as commutative diagrams, we do not require strict commutativity of the latter anymore, but only that the composites in the diagram are homotopic (and the homotopy between them is part of the data). Our main focus will be on the algebraic structure of an associative, commutative, and unital multiplication, as described above. If the axioms are only required to be satisfied up to *coherent homotopies*, such a structure is encoded by the so-called $\mathcal{E}_\infty$-operads, which we will introduce in Subsection 1.2. Actually, there exist various models of the latter, but as we will see, they are all equivalent.

To spice things up, one can consider that everything is happening under the action of a fixed finite group G; we enter the world of *equivariant homotopy theory* (we study spaces with a G-action, G-equivariant maps, G-equivariant notions of operads, etc.).

In this context, Blumberg and Hill defined in [BH15] $\mathcal{N}_\infty$-operads, a more subtle analog to $\mathcal{E}_\infty$-operads. They do encode commutative unital multiplications in the equivariant world but also take into account the finer structure of the group G that acts, including the restriction of its action to its various subgroups. In particular, $\mathcal{N}_\infty$-operads encode the existence of certain *transfer maps* on their algebras (i.e. their concrete realizations). To give a rough idea of what these are, consider a space X with a G-action. Then, given subgroups $K \leq H \leq G$, there is an inclusion of spaces of fixed points $X^H \to X^K$. A transfer map is a map in the other direction. We discuss $\mathcal{N}_\infty$-operads in Subsection 1.3.

In contrast to the case of $\mathcal{E}_\infty$-operads, as a consequence of the increased complexity in the equivariant setting, there exist different homotopy types of $\mathcal{N}_\infty$-operads, depending on which transfer maps are required to exist. And thus, mathematicians could not help but try to classify them. As we will see in Section 2, Blumberg and Hill did the first major step in this direction by producing a fully faithful embedding of the homotopy category of $\mathcal{N}_\infty$-operads (i.e. a category of equivalence classes of $\mathcal{N}_\infty$-operads) into the partially ordered set (a.k.a. poset) of *indexing systems*. The latter are certain objects related to the subgroups of G. Shortly after, this embedding was proven to be an equivalence of categories. The poset of indexing systems was also replaced by a simpler one, that of *transfer systems* in the poset of subgroups of G. Transfer systems are certain well-behaved subposets, and form themselves a poset under inclusion. This equivalence of categories establishes a beautiful correspondence between homotopy-theoretic flavored objects on the one hand, and objects of a combinatorial nature on the other hand.

Contemplating such an equivalence, one may wonder whether particular classes of $\mathcal{N}_\infty$-operads correspond to distinguished classes of transfer systems. This question has been (partially) answered for two families of $\mathcal{N}_\infty$-operads that "arise in nature" ([BH15, p22]): little disks operads (by [Rub21b]) and linear isometries operads (by [Mac23]). This is where representation theory enters the picture, as both families of $\mathcal{N}_\infty$-operads are parametrized by certain real representations of G. In the case of finite cyclic groups, these results give very simple combinatorial characterizations of the transfer systems corresponding to such $\mathcal{N}_\infty$-operads. For groups of small order, it is even easy to draw all possibilities by hand (many examples can be found in [Rub21b]). All of this will be discussed in Section 3.

This is not a research article but rather a survey designed to provide an elementary introduction to the subject. Therefore, we ignore many details related to working in the setting of model categories or ∞-categories rather than usual categories. Our goal today is to embark on a little journey where equivariant homotopy theory meets combinatorics and representation theory, guided by the story of the classification of $\mathcal{N}_\infty$-operads. We hope the reader will appreciate the beauty of the connections exhibited by the results we will state, and will take the time to read the technical details if they wish to pursue the adventure. In this case, we provide in Section 4 some references to study the subject, and list of several recent developments and open problems in the field.

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1. What are $\mathcal{N}_\infty$-operads?

In this section, we introduce the setup. We first define operads in general and their algebras. We then specialize to the case of $\mathcal{E}_\infty$-operads. Finally, we consider the G-equivariant setting for G a (finite) fixed group, and define $\mathcal{N}_\infty$-operads.

1.1. Topologists' point of view on operations

In the introduction, *operads* were advertised as abstract devices encoding algebraic structures. We will now give May's original definition of an operad. It contains a lot of technical axioms, but one should not be afraid of those, as they encode very natural properties to ask for. We give a simple way to interpret them right after the definition. In the sequel, we fix a symmetric monoidal category $(C, \otimes, \mathbb{1}_C)$ (readers unfamiliar with this notion may just think about the rough definition given in the introduction). We will mainly be interested in two examples: the category of topological spaces, where $\otimes$ is the usual Cartesian product and $\mathbb{1}_C$ is the one-point space, and the category of G-spaces, i.e. topological spaces with a G-action, where G is a fixed finite group, the functor $\otimes$ is again the product and $\mathbb{1}_C$ is the one-point space with the trivial G-action.

Definition 1.1. An *operad* $\mathcal{O}$ over C is a sequence $(\mathcal{O}(n))_{n\in\mathbb{N}}$ of objects in C, where each $\mathcal{O}(n)$ is endowed with an action of the symmetric group S_n, together with a unit $\mathbb{1}_\mathcal{O} : \mathbb{1}_C \to \mathcal{O}(1)$, and maps $\circ_i : \mathcal{O}(n) \times \mathcal{O}(m) \to \mathcal{O}(m+n-1)$ called *composition maps*, satisfying:

- **Associativity:** For all $n, m, k \in \mathbb{N}$, $j \leq n$ and $i \leq m+n-1$, we require:

$$(\circ_i) \circ (\circ_j \otimes \mathrm{id}) = \begin{cases} (\circ_{j+k-1}) \circ (\circ_i \otimes \mathrm{id}) \circ (\mathrm{id} \otimes B_{\mathcal{O}_m, \mathcal{O}_k}) & \text{if } 1 \leq i \leq j-1 \\ (\circ_j) \circ (\mathrm{id} \otimes \circ_{i-j+1}) & \text{if } j \leq i \leq m+j-1 \\ (\circ_j) \circ (\circ_{i-m+1} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes B_{\mathcal{O}_m, \mathcal{O}_k}) & \text{if } m+j \leq i \end{cases}$$

  where B is the braiding in $(C, \otimes, \mathbb{1}_C)$ (the natural isomorphism between $\otimes$ and its precomposition by the swap map, witnessing the commutativity of $\otimes$). For example, the first case amounts to the commutativity of the diagram

$$\begin{tikzcd} \mathcal{O}(n) \otimes \mathcal{O}(m) \otimes \mathcal{O}(k) \arrow[r, "\mathrm{id} \otimes B"] \arrow[d, "\circ_j \otimes \mathrm{id}"'] & \mathcal{O}(n) \otimes \mathcal{O}(k) \otimes \mathcal{O}(m) \arrow[r, "\circ_i \otimes \mathrm{id}"] & \mathcal{O}(n+k-1) \otimes \mathcal{O}(m) \arrow[d, "\circ_{j+k-1}"] \\ \mathcal{O}(n+m-1) \otimes \mathcal{O}(k) \arrow[rr, "\circ_i"'] & & \mathcal{O}(n+m+k-1). \end{tikzcd}$$

- **Equivariance:** For all $n, m \in \mathbb{N}$, $i \leq n$, $\sigma \in S_n$ and $\tau \in S_m$, the diagram

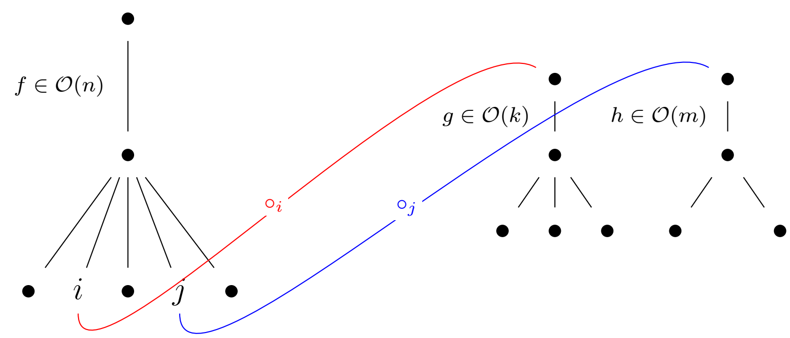
$$\begin{tikzcd} \mathcal{O}(n) \otimes \mathcal{O}(m) \arrow[r, "\circ_i"] \arrow[d, "\sigma \otimes \tau"'] & \mathcal{O}(n+m-1) \arrow[d, "(\sigma \circ_i \tau)"] \\ \mathcal{O}(n) \otimes \mathcal{O}(m) \arrow[r, "\circ_i"'] & \mathcal{O}(n+m-1) \end{tikzcd}$$

  must commute. Here $\sigma \circ_i \tau \in S_{n+m-1}$ is obtained by replacing the "i" entry in σ by τ, when both permutations are written as sequences $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ and $(\tau(1), \tau(2), \ldots, \tau(m))$, and shifting the indices in τ by $i-1$, and the entries $j > i$ in σ by $m-1$. For instance, if $n = 3$, $m = 7$, we have $(1, 3, 2) \circ_2 (4, 5) = (1, 9, 5, 6)$.

- **Unit:** the composition $\mathcal{O}(n) \cong \mathbb{1}_C \otimes \mathcal{O}(n) \xrightarrow{\mathbb{1}_{\mathcal{O}} \otimes \mathrm{id}} \mathcal{O}(1) \otimes \mathcal{O}(n) \xrightarrow{\circ_1} \mathcal{O}(n)$ is the identity, and similarly for $\mathcal{O}(n) \otimes \mathbb{1}_C$.

The intuition behind the definition is the following: thinking of $\mathcal{O}$ as representing an abstract algebraic structure, $\mathcal{O}(n)$ represents the collection of operations in arity n (i.e. with n inputs) needed to define this structure, the functor $\mathbb{1}_{\mathcal{O}}$ picks the identity operation in $\mathcal{O}(1)$, and the compositions $\circ_i$ correspond to plugging in the output of the second operation into the i-th input of the first one.

If one represents an n-ary operation by a tree, with n inputs branches at the bottom and one output branch at the top, then composition corresponds to tree grafting. We can then illustrate the associativity axiom in the case $i < j$ as follows:



The axiom says that the order in which we graft g and h does not matter. Indeed, $i < j$ means we are not grafting g onto the new inputs of $f \circ_j h$ corresponding to h, or vice versa. The other axioms have similar interpretations in terms of trees, and the formulas simply ensure that the intuition one obtains from trees is correct.

Example 1.2. Let V be vector space over a field k. An example of an operad over the symmetric monoidal category of k-vector spaces (with the usual tensor product and unit k) is the *endomorphism operad* End_V. For all $n \in \mathbb{N}$, $\mathrm{End}_V(n)$ is defined as $\mathrm{Hom}(V^{\otimes n}, V)$ the vector space of linear maps $V^{\otimes n} \to V$ (multilinear maps). Then the symmetric group S_n acts by permutations on $V^{\otimes n}$ (that is, for $f : V^{\otimes n} \to V$ and $\sigma \in S_n$, let $(\sigma \cdot f)(v_1 \otimes \cdots \otimes v_n) = f(v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)})$). The composition $f \circ_i g$ of two multilinear maps corresponds to plugging in the result of the second one $g(\cdot, \ldots, \cdot)$ into the i-th entry of the first one $f(\cdot, \ldots, \cdot)$, to obtain a new map $f(\cdot, \ldots, \cdot, g(\cdot, \ldots, \cdot), \cdot, \ldots, \cdot)$. The endomorphism operad can be defined in any symmetric monoidal category C with a suitable notion of Hom-functor, in particular for a subcategory of “nice” topological spaces (with $\mathrm{Hom}(X, Y)$ the set of continuous maps $X \to Y$ with the compact-open topology).

Definition 1.3. A *morphism of operads* between $\mathcal{O}$ and $\mathcal{O}'$ is a collection of morphisms $\{f_n : \mathcal{O}(n) \to \mathcal{O}'(n)\}_{n \in \mathbb{N}}$ in C, where f_n is S_n-equivariant for all $n \in \mathbb{N}$, and these morphisms are compatible with the units and the compositions (i.e. $f_1 \circ \mathbb{1}_{\mathcal{O}} = \mathbb{1}'_{\mathcal{O}}$ and $f_{m+n-1} \circ (\circ_i^{\mathcal{O}}) = (\circ_i^{\mathcal{O}'}) \circ (f_n \otimes f_m)$ for all $n, m \in \mathbb{N}$ and $1 \leq i \leq n$).

Operads encode abstract operations and the axioms they satisfy. Their concrete realizations, i.e. objects endowed with the data of such operations, are called *algebras* over this operad. The slogan is: “Algebras are to operads what representations are to groups”. If C is a category of spaces or sets, the idea is to associate to each *element* of $\mathcal{O}(n)$ an n-ary operation on a space or set A, i.e. a map $A^n \to A$.

Definition 1.4. Assume that the endomorphism operad can be defined in C (see Example 1.2). Let $\mathcal{O}$ be an operad over C. An *algebra over* $\mathcal{O}$, or $\mathcal{O}$-*algebra*, is an object $A \in C$ endowed with a morphism of operads $\mathcal{O} \to \mathrm{End}_A$. Equivalently, it is an object $A \in C$ with the data of a family of morphisms $(\gamma_n : \mathcal{O}(n) \otimes A^{\otimes n} \to A)_{n\in\mathbb{N}}$ in C, compatible with $\mathbb{1}_{\mathcal{O}}$ and $\circ_i$, such that γ_n is S_n-equivariant (when A has the trivial action and $A^{\otimes n}$ the action by permutations).

Example 1.5. Let Comm be the operad given by the constant sequence $(\mathbb{1}_C)_{n\in\mathbb{N}}$, with trivial composition maps and S_n-action. Then, the Comm-algebras are exactly the commutative monoid objects in C. Indeed, let $M \in C$ be a Comm-algebra with structure maps $\{\gamma_n\}_{n\in\mathbb{N}}$. Then $\mu := \gamma_2 : \mathbb{1}_C \otimes M \otimes M \cong M \otimes M \to M$ and $\eta := \gamma_0 : \mathbb{1}_C \to M$ define the multiplication and the unit of a commutative monoid structure on M. For example, note that by S_n-equivariance, we have $\gamma_2(\mathrm{id} \otimes B) = \gamma_2$, i.e. $\mu B = \mu$, where B is the braiding of C (the swap map on $M \otimes M$), so μ is commutative. Conversely, given a commutative monoid (M, μ, η) in C, we have maps $\mathrm{Comm}(n) \otimes M^{\otimes n} \cong M^{\otimes n} \to M$ by iterating the multiplications μ when $n > 0$ (order does not matter, by the associativity and commutativity of $\otimes$ and μ) and η when $n = 0$. This defines a Comm-algebra structure on M. In particular, if C is the category of small categories, with $\otimes$ the product and $\mathbb{1}_C$ the one-point category, then a Comm-algebra in C is a symmetric monoidal category.

Example 1.6. The simplex operad Simp is an operad over the symmetric monoidal category of (nice) topological spaces $(\mathrm{Top}, \times, \{*\})$. Let $\mathrm{Simp}(n)$ be the standard $(n-1)$-simplex, i.e. $\left\{(x_0, \dots, x_{n-1}) \in \mathbb{R}^n \mid x_i \geq 0, \sum_{i=0}^{n-1} x_i = 1\right\}$. The compositions are the maps

$$\begin{aligned} \circ_i : \mathrm{Simp}(n) \times \mathrm{Simp}(m) &\longrightarrow \mathrm{Simp}(m+n-1) \\ (\overline{x}, \overline{y}) &\longmapsto (\overline{x}_0, \dots, \overline{x}_{i-2}, \overline{x}_{i-1}\overline{y}_0, \dots, \overline{x}_{i-1}\overline{y}_{m-1}, \overline{x}_i, \dots, \overline{x}_{n-1}). \end{aligned}$$

The group S_n acts by permutation of the coordinates. Any convex subset $X \subseteq \mathbb{R}^m$ has a Simp-algebra structure, given by convex combinations (to each $\overline{x} \in \mathrm{Simp}(n)$ we associate the map $X^n \longrightarrow X$, $(v_0, \dots, v_{n-1}) \longmapsto \sum_{i=0}^{n-1} \overline{x}_i v_i$). In particular, the half-line $\mathbb{R}_{\geq 0} := [0, +\infty[$ is a Simp-algebra. Viewing $\mathbb{R}_{\geq 0}$ as a single object category with one morphism for each element in $\mathbb{R}_{\geq 0}$ (composition is given by addition), we can consider *internal* Simp-algebras. Roughly speaking, the latter are like Simp-algebras, but in this new category $\mathbb{R}_{\geq 0}$. They are also defined as a collection of n-ary operations. Fun fact: they correspond exactly to the constant multiples of the Shannon entropy, and further interesting connections can be made (see [Lei11]).

1.2. Relaxing the relations between the operations : $\mathcal{E}_\infty$-operads

If $\mathcal{O}$ is an operad, the data of an $\mathcal{O}$-algebra structure on a given object involves the data of structure maps satisfying certain relations, which can be written in the form of equalities of various compositions of morphisms. A general principle in homotopy theory is to relax this equality condition and ask instead for homotopies ("continuous deformations") between the maps involved (and also remember the data of these homotopies). Applying this to the operad Comm from Example 1.5, we are looking for a notion of a multiplication that is unital, commutative, and associative up to homotopy only. For instance, instead of requiring that the multiplication map is *isomorphic* to its precomposition by the swap map, we only require them to be homotopy equivalent. This notion is encoded by $\mathcal{E}_\infty$-operads.

Definition 1.7 ([May72])**.** An $\mathcal{E}_\infty$*-operad* is an operad $\mathcal{O}$ over Top such that for all $n \in \mathbb{N}$, $\mathcal{O}(n)$ is contractible (homotopy equivalent to a point) and the action of S_n on it is free. An $\mathcal{E}_\infty$*-space* is an algebra over any $\mathcal{E}_\infty$-operad over Top.

Viewing $\mathcal{O}(n)$ as a "space of n-ary operations", the contractibility stands for the fact that up to homotopy, we have essentially one single operation in this arity, representing the multiplication of n elements in any order.

Remark 1.8. Any Comm-algebra in Top (see Example 1.5) is also an $\mathcal{E}_\infty$-space. Indeed, intuitively, a Comm-algebra structure encodes a *strictly* associative and commutative multiplication, which is, in particular, a *homotopy* associative and commutative multiplication, and thus induces the structure of an $\mathcal{E}_\infty$-algebra.

All $\mathcal{E}_\infty$-operads are equivalent (in a sense to be made precise) as topological operads to the operad Comm of Example 1.5. However, the added requirement that the actions of the symmetric groups are free makes $\mathcal{E}_\infty$-operads into a special class of "replacements" for the Comm-operad, sometimes called Σ*-cofibrant resolutions* of the latter. In this sense, there is only one equivalence class of $\mathcal{E}_\infty$-operads. We have a stronger result:

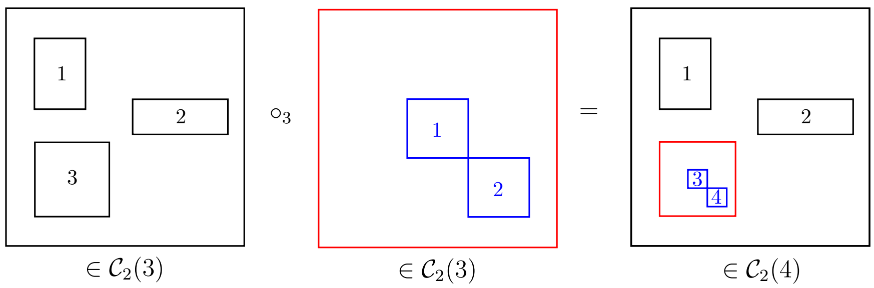
Theorem 1.9 ([May72])**.** *If $\mathcal{E}$ and $\mathcal{E}'$ are two $\mathcal{E}_\infty$-operads, the (homotopy) categories of $\mathcal{E}$-algebras and $\mathcal{E}'$-algebras are equivalent. In particular, the category of $\mathcal{E}_\infty$-spaces is well-defined.*

We may therefore choose our favorite model of $\mathcal{E}_\infty$-operad. The most common one is probably the operad of little cubes. It is obtained as the infinite "union" (actually, colimit) over $k \geq 1$ of the operads of little k-cubes, which we now define.

Definition 1.10. Let $k \in \mathbb{N}^+ := \{1, 2, \dots\}$. The *operad of little k-cubes* $\mathcal{C}_k$ over Top is given by $\mathcal{C}_k(0) = \{*\}$ and for $n \geq 1$, $\mathcal{C}_k(n)$ is the space of n-uples of disjoint rectilinear embeddings of the k-dimensional unit cube into itself. Explicitly, an element of $\mathcal{C}_k(n)$ is a n-uple $(f_1, \dots, f_n)$ of maps $f_i : I^k \to I^k$ where $I = [0, 1]$ is the unit interval, such that $f_i(\mathrm{int}(I^k)) \cap f_j(\mathrm{int}(I^k)) = \emptyset \ \forall i \neq j$, and for all $i \leq n$, f_i is of the form $\overline{x} \mapsto (a_{i,1}\overline{x}_1 + b_{i,1}, \ldots, a_{i,n}\overline{x}_n + b_{i,n})$ with $a_{i,j} \in \mathbb{R}_{>0}$, $b_{i,j} \in \mathbb{R}_{\geq 0}$. This space is endowed with the Euclidean topology when we view the coordinates $a_{i,j}$ and $b_{i,j}$ in $\mathbb{R}^{2nk}$. Then S_n acts by permutation of the embeddings in an n-uple. Let

$$\begin{aligned} \circ_i \ : \ \mathcal{C}_k(n) \times \mathcal{C}_k(m) &\longrightarrow \mathcal{C}_k(m+n-1) \\ ((f_1, \ldots, f_n), (g_1, \ldots, g_m)) &\longmapsto (f_1, \ldots, f_{i-1}, f_i \circ g_1, \ldots, f_i \circ g_m, f_{i+1}, \ldots, f_n). \end{aligned}$$

An element of $\mathcal{C}_k(n)$ can be viewed as a k-cube with n smaller ones inside, labeled from 1 to n. Here is an example of composition in the case $k = 2$:



Definition 1.11. The *operad of little cubes* $\mathcal{C}_\infty$ is given by $\mathcal{C}_\infty(n) = \mathrm{colim}_{k\in\mathbb{N}}\, \mathcal{C}_k(n)$ for all $n \in \mathbb{N}$. Readers unfamiliar with category theory may view the colimit "colim" as a union, where $\mathcal{C}_k(n)$ is viewed as a subobject of $\mathcal{C}_{k+1}(n)$ via the map adding the identity on the last component of each embedding (a k-uple of maps $(f_1, \ldots, f_k)$ is sent to $(f_1 \times \mathrm{id}_I, \ldots, f_k \times \mathrm{id}_I)$). For instance, for $k = 1$, this corresponds to taking the vertical strips corresponding to each interval.

Proposition 1.12 ([May72])**.** *The operad $\mathcal{C}_\infty$ is an $\mathcal{E}_\infty$-operad.*

Sketch of proof. The action of S_n on $\mathcal{C}_k(n)$ for some $k, n \in \mathbb{N}$ being by relabeling the little k-cubes, it is free. Since the maps $\mathcal{C}_k(n) \to \mathcal{C}_{k+1}(n)$ are actual embeddings, their colimit can be expressed as an infinite union, and thus the action of S_n on $\mathcal{C}_\infty(n)$ is also free. To prove that $\mathcal{C}_\infty(n)$ is contractible, May shows that $\mathcal{C}_k(j)$ is homotopy equivalent to $F(\mathbb{R}^k, j)$ the j-th configuration space in $\mathbb{R}^k$ (the space of j-uples of distinct points in $\mathbb{R}^k$). The latter is not contractible, but as k goes to infinity, its connectivity increases (i.e. more and more homotopy groups in low degrees vanish). Passing to the colimit, one obtains that the homotopy groups of $\mathcal{C}_\infty(n)$ are all trivial (it is weakly contractible). One can show that $\mathcal{C}_\infty(n)$ is homotopy equivalent to a CW-complex, therefore it is also contractible. $\square$

May's recognition theorem is probably the most famous use of $\mathcal{E}_\infty$-operads.

Theorem 1.13 ([May72])**.** *For $1 \leq n \leq \infty$, a connected space X is an n-fold loop space (see Definition 1.15) if and only if it admits the structure of a $\mathcal{C}_n$-algebra.*

Remark 1.14. In the case $n = \infty$, by Proposition 1.12, $\mathcal{C}_\infty$ is an $\mathcal{E}_\infty$-operad. We have also seen that all $\mathcal{E}_\infty$-operads are equivalent. Thus, we may replace "$\mathcal{C}_\infty$-algebra" in the theorem with "$\mathcal{E}_\infty$-space". In particular, since connected commutative monoids in Top (for example, the circle $\mathbb{S}^1$) are Comm-algebras by Remark 1.8, they are also $\mathcal{E}_\infty$-algebras and thus are infinite loop spaces.

Definition 1.15. The loop space functor $\Omega : \mathrm{Top}_* \to \mathrm{Top}_*$ sends a pointed space X to the space of continuous pointed maps $\mathbb{S}^1 \to X$ endowed with the compact open topology. Then $\Omega^k X$ can be viewed as the space of pointed maps $\mathbb{S}^k \to X$. We say that a pointed space X is an *n-fold loop space* for $n \in \mathbb{N}^+$ if there exists a pointed space Y such that X is weakly equivalent to $\Omega^n Y$, and that X is an *infinite loop space* if there exists a sequence of pointed spaces $(Y_n)_{n\in\mathbb{N}}$ with $Y_0 = X$ and pointed weak equivalences $Y_n \xrightarrow{\simeq} \Omega Y_{n+1}$ for all $n \in \mathbb{N}$.

Remark 1.16. The "only if" direction in Theorem 1.13 is not too hard to prove (and the converse is much harder). Indeed, if (X, x_0) is a pointed space, to define a $\mathcal{C}_k$-algebra structure on $\Omega^k X$, we need maps $\mathcal{C}_k(n) \times (\Omega^k X)^n \to \Omega^k X$. Given $(f, g) = ((f_i)_{i\leq n}, (g_i)_{i\leq n}) \in \mathcal{C}_k(n) \times (\Omega^k X)^n$, consider the composition

$$I^k \longrightarrow I^k/(I^k \setminus (f_1(\mathring{I}^k) \cup \cdots \cup f_n(\mathring{I}^k))) \xrightarrow{\sim} \bigvee_{i=1}^{n} \mathbb{S}^k \xrightarrow{\bigvee_{i=1}^n g_i} X.$$

This induces a map $\mathbb{S}^k \to X$ by collapsing the boundary in I^k. If $k = 1$, for little intervals $[a_1, b_1], \ldots [a_n, b_n]$, we obtain the following concatenation of the loops $g_1, \ldots, g_n$: setting $b_0 = 0$, from time b_{i-1} to a_i we stay at the base point, and from time a_i to b_i we follow the (reparameterization of the) loop g_i, for all $i \geq 1$. Since concatenation of loops is associative and reparameterization-invariant up to homotopy, one might want to identify all n-uples of embeddings with the one giving a regular partition of I into n intervals. We would then only have $n!$ points in each arity; one for each labeling of the intervals. This is called the Assoc operad and is simply the non-commutative version of Comm. But the operad $\mathcal{C}_1$ has more data. As k increases, there are more and more levels of "homotopies between homotopies etc" relating the different ways of permuting the little k-cubes embedded inside the bigger one, encoding "more and more commutativity up to homotopy". This is the whole point of having an $\mathcal{E}_\infty$ operad: it is equivalent to Comm but encodes the properties *up to homotopy* only.

1.3. When group actions enter the picture: $\mathcal{N}_\infty$-operads

From now on, unless stated otherwise, G is a fixed finite group. In [BH15], Blumberg and Hill generalize the theory of $\mathcal{E}_\infty$-operads to the G-equivariant setting, namely where everything is considered under the action of G. A first way of generalizing Definition 1.7 would simply be to replace the category of spaces, Top, with that of G-spaces, Top_G. Instead, they define the richer notion of $\mathcal{N}_\infty$-operad, which takes into account the structure of G and its action at a finer level, as we will now see.

Definition 1.17. An $\mathcal{N}_\infty$-*operad* is an operad $\mathcal{O}$ over Top_G such that:

1. For all $n \in \mathbb{N}$, the $(G \times S_n)$-action on $\mathcal{O}(n)$ admits a G-fixed point, and the restriction of the action to S_n is free.
2. For any subgroup $\Gamma \leq G \times S_n$, the space $\mathcal{O}(n)^\Gamma$ of Γ-fixed points is either empty or contractible (as a topological space, not necessarily equivariantly).

Remark 1.18. In particular, the composition maps and unit for $\mathcal{O}$ are G-equivariant.

Since an $\mathcal{N}_\infty$-operad is more than just an $\mathcal{E}_\infty$-operad in G-spaces, its algebras are endowed with additional structure, namely *transfer maps*. Which transfers are supplied is parametrized by the data of which spaces of fixed points in the operad are contractible, rather than empty. This data distinguishes particular inclusions in the poset of subgroups of G, i.e. maps in the poset, called *admissible relations*.

Definition 1.19. Let $H \leq G$ be a subgroup and T a finite H-set. Choose an enumeration of T as $\{t_1, \dots, t_{|T|}\}$. The *graph subgroup associated with* T is (the conjugacy class of) the subgroup $\Gamma_T \leq G \times S_{|T|}$ given by the graph of the morphism $H \to S_{|T|}$, $h \mapsto \sigma_h$, such that $h \cdot t_i = t_{\sigma_h(i)}$ for all $i \leq |T|$. If $T = H/K$ for some $K \leq H$, with H acting by multiplication on the left, we write $\Gamma_T = \Gamma_{H,K}$. This is well-defined under changing the enumeration of T because any relabeling of T yields a subgroup conjugate to Γ_T.

Definition 1.20. Let $\mathcal{O}$ be an $\mathcal{N}_\infty$-operad and $H \leq G$. A finite H-set T is called *admissible* if $\mathcal{O}(|T|)^{\Gamma_T} \neq \emptyset$ (this only depends on the conjugacy class of Γ_T). Given $K \leq H \leq G$, the relation $K \leq H$, also viewed as a morphism from K to H in the poset of subgroups of G, is called *admissible* if H/K is an admissible H-set for $\mathcal{O}$.

Theorem 1.21 ([BH15], [Rub21b, Rmk 3.5])**.** *Let X be an algebra over an $\mathcal{N}_\infty$-operad $\mathcal{O}$, and let $K \leq H \leq G$ be subgroups, with $K \leq H$ an admissible relation for $\mathcal{O}$. Then, the $\mathcal{O}$-algebra structure of X provides contractible spaces of* internal transfer maps *of H-spaces $X^K \longrightarrow X^H$ and* external transfer maps *of G-spaces $G \times_H X^{\times H/K} \longrightarrow X$.*

Remark 1.22. The space of all maps $X^K \to X^H$ is not contractible in general. The above result only states that there are essentially unique choices of internal transfer maps supplied by the $\mathcal{O}$-algebra structure, and similarly for the external transfers.

Weak equivalences between $\mathcal{N}_\infty$-operads should also respect the finer structure of all spaces of fixed points. Otherwise, they would amount only to weak equivalences between the underlying topological $\mathcal{E}_\infty$-operads. But all categories of algebras over such operads are equivalent, whereas, for different $\mathcal{N}_\infty$-operads, the algebras do not look quite the same due to the existence of various transfer maps.

Definition 1.23. A morphism of operads in G-spaces $f : \mathcal{O} \to \mathcal{O}'$ between two $\mathcal{N}_\infty$-operads is an *equivalence* if the induced maps $f^\Gamma : \mathcal{O}(n)^\Gamma \to \mathcal{O}'(n)^\Gamma$ are weak homotopy equivalences between the underlying topological spaces for all subgroups $\Gamma \leq G \times S_n$ and $n \in \mathbb{N}$.

This notion of equivalence is sensible because equivalent $\mathcal{N}_\infty$-operads $\mathcal{O}$ and $\mathcal{O}'$ have equivalent categories of algebras (provided that their G-spaces of operations $\mathcal{O}(n)$ and $\mathcal{O}'(n)$ are nice enough for all $n \in \mathbb{N}$, see [BH15, Thm A.3]).

2. Classifying $\mathcal{N}_\infty$-operads by combinatorial objects

We briefly mentioned in the previous subsection the existence of several non-equivalent classes of $\mathcal{N}_\infty$-operads. By formally inverting the equivalences in the category of $\mathcal{N}_\infty$-operads, one obtains the *homotopy category* of $\mathcal{N}_\infty$-operads, i.e. a category of equivalence classes of $\mathcal{N}_\infty$-operads. Then, "classifying $\mathcal{N}_\infty$-operads up to homotopy" can be understood as "finding an equivalent explicit description of the homotopy category of $\mathcal{N}_\infty$-operads". This is what this section is dedicated to.

In [BH15], Blumberg and Hill partially classified $\mathcal{N}_\infty$-operads by constructing a fully faithful embedding of the homotopy category of $\mathcal{N}_\infty$-operads into the poset category of *indexing systems*. The latter are certain posets expressed in terms of categories of H-sets when H varies among the subgroups of G. The classification was independently completed by Bonventre and Pereira ([BP21]), Gutiérrez and White ([GW18]), and Rubin ([Rub21a]), who proved that this embedding was an equivalence of categories. A refined version was proven in [Rub21b] and [BBR21]. More precisely, the poset of indexing systems was replaced by the equivalent poset of *transfer systems*. The latter can be seen as *generating data* for the former; they are thus "smaller" objects, easier to deal with. In this subsection, we will focus on the approach in [Rub21b] because it is more combinatorial and elementary.

Definition 2.1. A *transfer system on G* is a subposet $(\mathcal{T}, \to)$ of the poset of subgroups of G with respect to inclusion $(\mathrm{Sub}(G), \leq)$, such that

1. It contains all the subgroups of G.
2. It is closed under conjugation: if $K \to H$ then $gKg^{-1} \to gHg^{-1}$ for all $g \in G$.
3. It is closed under restriction: if $K \to H$ and $M \leq H$ then $K \cap M \to M$.

The set of transfer systems on G forms itself a poset $\mathrm{Tr}(G)$ with respect to inclusion.

When G is abelian, condition 2 holds trivially, and the definition no longer uses the group structure. It becomes a purely combinatorial notion that can be generalized to any lattice (replacing the intersection by the meet operation).

Rubin uses in his proof a discrete analog of $\mathcal{N}_\infty$-operads.

Definition 2.2. An N-operad $\mathcal{O}$ is an operad in G-sets such that the action of S_n on $\mathcal{O}(n)$ is free for all $n \in \mathbb{N}$, and the fixed points $\mathcal{O}(0)^G$ and $\mathcal{O}(2)^G$ are non-empty.

N-operads are a good model for $\mathcal{N}_\infty$-operads, in the sense that their homotopy category is equivalent to that of $\mathcal{N}_\infty$-operads, and one can prove that they have good properties with respect to the functor $\underline{\mathcal{C}}$ from Theorem 2.3 (see [Rub19a]).

Theorem 2.3 ([BH15, Thm 3.24], and e.g. [Rub21a])**.** *There is an equivalence*

$$\underline{\mathcal{C}} : Ho(\mathcal{N}_\infty\text{-}Op) \longrightarrow Tr(G)$$

between the homotopy category of $\mathcal{N}_\infty$-operads and the poset of transfer systems on G. The functor $\underline{\mathcal{C}}$ sends an $\mathcal{N}_\infty$-operad $\mathcal{O}$ to the transfer system $\to$ where $K \to H$ if and only $K \leq H$ and the relation $K \leq H$ is admissible for $\mathcal{O}$ (Definition 1.20).

Sketch of proof. One first shows that the functor $\underline{\mathcal{C}}$ is well-defined, i.e. takes values in transfer systems on G and sends equivalent $\mathcal{N}_\infty$-operads to the same transfer system. The latter follows from Definition 1.23. The former is longish but easy to check by chasing the axioms in Definitions 1.17 and 2.1; one has to define certain explicit fixed points by hand. Now, to show that $\underline{\mathcal{C}}$ is an equivalence of categories, we show it is fully faithful and essentially surjective.

Step 1: the functor $\underline{\mathcal{C}}$ is full. Assume $\mathcal{T} = \underline{\mathcal{C}}(\mathcal{N})$ and $\mathcal{T}' = \underline{\mathcal{C}}(\mathcal{N}')$ for $\mathcal{N}_\infty$-operads $\mathcal{N}, \mathcal{N}'$, and there is a map from $\mathcal{T}$ to $\mathcal{T}'$ in the poset Tr(G), i.e. $\mathcal{T} \subseteq \mathcal{T}'$. We have to show that there is a map $\mathcal{N} \to \mathcal{N}'$ in the homotopy category. This uses a trick of May, also used in [BH15]: consider the zigzag of maps $\mathcal{N} \leftarrow \mathcal{N} \times \mathcal{N}' \to \mathcal{N}'$.

We have $\underline{\mathcal{C}}(\mathcal{N} \times \mathcal{N}') = \underline{\mathcal{C}}(\mathcal{N}) \cap \underline{\mathcal{C}}(\mathcal{N}')$. Indeed, the relation $K \leq H$ is admissible for the product $\mathcal{N} \times \mathcal{N}'$ if and only if

$$\begin{aligned} ((\mathcal{N} \times \mathcal{N}')([H:K]))^{\Gamma_{H,K}} &= (\mathcal{N}([H:K]) \times \mathcal{N}'([H:K]))^{\Gamma_{H,K}} \\ &= (\mathcal{N}([H:K]))^{\Gamma_{H,K}} \times (\mathcal{N}'([H:K]))^{\Gamma_{H,K}} \neq \emptyset, \end{aligned}$$

i.e. if and only if both factors are non-empty. This happens exactly when H/K is admissible for both $\mathcal{N}$ and $\mathcal{N}'$, i.e. when $K \leq H$ is in both $\mathcal{T}$ and $\mathcal{T}'$.

We now claim that the condition $\mathcal{T} \subseteq \mathcal{T}'$ implies that $\mathcal{N} \times \mathcal{N}' \to \mathcal{N}$ is a weak equivalence. To prove this, it suffices to check that the fixed points of the left-hand side are contractible if and only if they are contractible on the right-hand side, since a map between contractible spaces is necessarily a weak equivalence. Let $n \in \mathbb{N}$ and $\Gamma \leq G \times S_n$. If Γ intersects S_n non-trivially, the fixed points on both sides are empty since S_n acts freely. If $\Gamma \cap S_n = \{0\}$, it is easy to prove that Γ is a graph subgroup. Then, if $(\mathcal{N} \times \mathcal{N}')(n)^\Gamma = \mathcal{N}(n)^\Gamma \times \mathcal{N}'(n)^\Gamma \neq \emptyset$, in particular $\mathcal{N}(n)^\Gamma$ is non-empty, so it is contractible. Conversely, if $\mathcal{N}(n)^\Gamma \neq \emptyset$, let H be the projection of Γ onto G. Then the H-set structure on $\{1, \ldots, n\}$ associated with Γ makes each orbit (which is isomorphic as an H-set to H/K_i for some $K_i \leq H$, $i \leq k$) an admissible set for $\mathcal{N}$, and hence for $\mathcal{N}'$ too, since $\mathcal{T} \subseteq \mathcal{T}'$. Therefore, for all $i \leq k$, there exists $x_i \in \mathcal{N}'([H : K_i])^{\Gamma_{H,K_i}} \neq \emptyset$. Let $y \in \mathcal{N}'(k)^G$. Plugging in the operation x_i in the i-th input of y for all $i \leq k$, we obtain a new operation $z \in \mathcal{N}'(n)$, since $[H : K_1] + \cdots + [H : K_k] = n$. Furthermore, z is a Γ-fixed point because the projection of Γ on S_n consists only of shuffles within the orbits, by definition. Thus $\mathcal{N}'(n)^\Gamma \neq \emptyset$ and $(\mathcal{N} \times \mathcal{N}')(n)^\Gamma \neq \emptyset$, as desired.

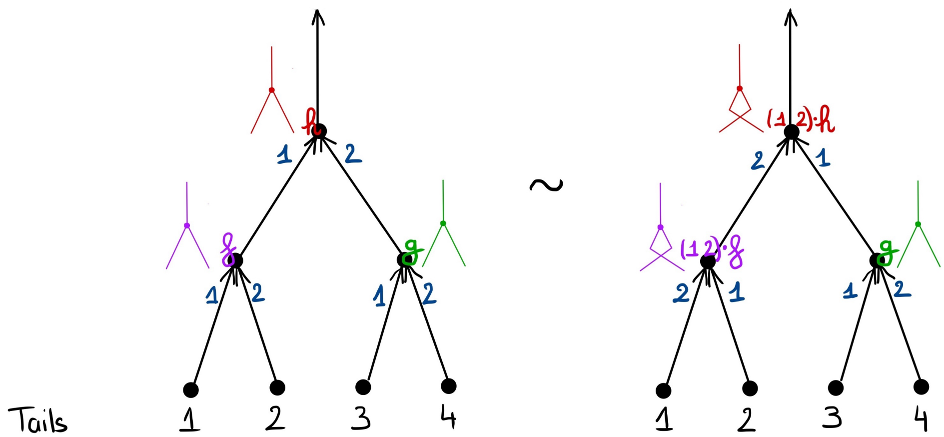
Since weak equivalences are invertible in the homotopy category, the zigzag of maps mentioned above becomes an actual map $\mathcal{N} \to \mathcal{N}'$, as needed.

Step 2: $\underline{\mathcal{C}}$ is faithful. Since the target is a poset, it suffices to show that every class of morphisms between given objects in $\mathrm{Ho}(\mathcal{N}_\infty\text{-Op})$ is empty or consists of a single point. We skip this harder part of the proof, which uses a structure of model category on $\mathcal{N}_\infty$-Op to have an explicit description of the homotopy category.

Step 3: $\underline{\mathcal{C}}$ is essentially surjective. Let $\mathcal{T}$ be a transfer system on G. We want to build an $\mathcal{N}_\infty$-operad $\mathcal{N}$ with $\underline{\mathcal{C}}(\mathcal{N}) = \mathcal{T}$. It will be constructed as a classifying space of a free N-operad (Definition 2.2).

Firstly, for all $n \in \mathbb{N}$, let $\widetilde{S}_\mathcal{T}(n)$ be the G-set $\coprod_{(K \to H) \in \mathcal{T},\ [H:K]=n} (G \times S_n)/\Gamma_{H,K}$. Let $S_\mathcal{T}(n) = \tilde{S}_\mathcal{T}(n)$ if $n \neq 0, 2$, or $S_\mathcal{T}(n) = \tilde{S}_\mathcal{T}(n) \amalg (G \times S_n)/(G \times \{\mathrm{id}\})$ if $n = 0, 2$.

Then, take the free operad $\mathbb{T}(S_\mathcal{T})$ in G-sets over the collection $(S_\mathcal{T}(n))_{n \in \mathbb{N}}$. It can be explicitly described as follows: $\mathbb{T}(S_\mathcal{T})(n)$ is the set of equivalence classes of n-trees, where each vertex is decorated by an element of $S_\mathcal{T}(n)$. Here, an n-tree (see the illustration below) is defined as a finite connected directed graph such that any vertex v has at most one outgoing edge, its ingoing edges are numbered (this labeling is part of the data), and there is one arrow with no vertex at the end, and n vertices without ingoing edges, called *tails*, numbered from 1 to n. The equivalence relation identifies trees where the same permutation has been applied to both the labels of the ingoing edges of a vertex and the operation decorating the vertex itself. For example, if $f, g, h \in S_{\mathcal{T}}(2)$, the following 2-trees with labeled vertices are equivalent:



Finally, turn $\mathbb{T}(S_{\mathcal{T}})$ into an operad in G-spaces by first taking the cofree category, then the nerve, and finally the geometric realization, namely the composition

$$E:\ \mathrm{Set} \underset{\text{forgetful}}{\overset{\text{cofree}}{\underset{\top}{\rightleftarrows}}} \mathrm{Cat} \underset{\text{realization}}{\overset{\text{nerve}}{\underset{\top}{\rightleftarrows}}} \mathrm{sSet} \underset{\text{singular set}}{\overset{|\cdot|}{\underset{\perp}{\rightleftarrows}}} \mathrm{Top}$$

Let $\mathcal{N} := E(\mathbb{T}(S_{\mathcal{T}}))$. We have to check that $\mathcal{N}$ is indeed an $\mathcal{N}_\infty$-operad, and that $\underline{\mathcal{C}}(\mathcal{N}) = \mathcal{T}$. For $\mathcal{N}$ to be an operad in G-spaces, one has to show that the functor E respects the symmetric monoidal structures, namely the products, the units, and the compatibilities between them. To check that $\mathcal{N}$ is $\mathcal{N}_\infty$, the crucial point is that the functor E "creates contractible spaces, by attaching cells to kill homotopy". Indeed, the cofree construction adds an edge between any (ordered) pair of points, the nerve takes all possible paths, and the geometric realization glues cells in every dimension "in all possible ways". In view of the definition of an $\mathcal{N}_\infty$-operad, one then has to study how E interacts with fixed points and homotopies, and one can reduce to showing the corresponding properties for $\mathbb{T}(S_{\mathcal{T}})$. More precisely, using the explicit description of the free operad, one shows the following:

Claim: The operad $\mathbb{T}(S_{\mathcal{T}})$ has a free action of the symmetric groups in each arity, a G-fixed unit, and $\mathbb{T}(S_{\mathcal{T}})(n)^{G\times\{\mathrm{id}\}} \neq \emptyset$ for all $n \in \mathbb{N}$. Moreover, for all $K \leq H \leq G$, we have $(K \leq H) \in \mathcal{T}$ if and only if $\mathbb{T}(S_{\mathcal{T}})([H:K])^{\Gamma_{H,K}} \neq \emptyset$. □

3. Even more combinatorics: particular cases of the classification

As mentioned in the introduction, one might try to restrict the equivalence of Theorem 2.3 to particular families of $\mathcal{N}_\infty$-operads, and then determine what are the corresponding transfer systems. In this section, we will consider two families of $\mathcal{N}_\infty$-operads for which the question has been answered in an interesting way.

3.1. **Little disks operads**

The first family of $\mathcal{N}_\infty$-operads we will consider is that of *little disks operads* (strongly inspired, as the name indicates, from the little cubes operad from Subsection 1.2). It is parametrized by a specific type of representations of G:

Definition 3.1. A *G-universe* $\mathcal{U}$ is a countably infinite-dimensional real representation of G of the form $\bigoplus_{\mathbb{N}}(\mathbb{R}_{\text{triv}} \oplus V_1 \oplus \cdots \oplus V_k)$, with $V_1, \ldots, V_k$ irreducible finite-dimensional real representations of G by linear isometries (we implicitly choose isomorphisms $V_i \cong \mathbb{R}^{\dim V_i}$, and this induces an inner product on each of the V_i's).

Definition 3.2. Let $\mathcal{U}$ be a G-universe. The operad of little disks $\mathcal{D}(\mathcal{U})$ is defined in arity $n \in \mathbb{N}$ by $\mathcal{D}(\mathcal{U})(n) = \bigcup_{V \leq \mathcal{U}} \mathcal{D}_V(\mathcal{U})(n)$, where the union (colimit, actually) is indexed by finite-dimensional subrepresentations $V \leq \mathcal{U}$, and $\mathcal{D}_V(\mathcal{U})(n)$ is the space of n-uples of affine embeddings from the unit disk $\mathbb{D}(V) \subseteq V$ into itself, with disjoint interiors. The composition maps and action of S_n are as in Definition 1.10, and G acts by conjugation on each embedding in an n-uple (given an embedding $f : \mathbb{D}(V) \to \mathbb{D}(V)$, let $(g \cdot f)(v) = g \cdot (f(g^{-1}v))$ for all $v \in \mathbb{D}(V)$ and $g \in G$).

Remark 3.3. As Blumberg and Hill write in [BH15], little disks operads do not have very good topological properties, so one usually considers a "thickening" of those, called *Steiner operads*. However, by [BH15, Prop. 3.13], given a G-universe $\mathcal{U}$, the associated Steiner and little disks operads are equivalent as $\mathcal{N}_\infty$-operads.

Theorem 3.4 ([Rub21b, Thm 4.11])**.** *Let G be a finite* abelian *group and $(\mathcal{T}, \to)$ a transfer system on G. Then $(\mathcal{T}, \to)$ corresponds to a little disks operad under the equivalence of Theorem 2.3 if and only if it is generated by relations of the form $H \leq G$ where G/H is cyclic.*

Remark 3.5. In the case of cyclic groups, the theorem simplifies further because any quotient by a subgroup is cyclic. Thus, a transfer system on a cyclic group is realized by a little disks operad if and only if it is generated by the relations with target G that it contains. Such a transfer system is called *cosaturated*.

To prove Theorem 3.4, we need a characterization of the transfer system associated with a little disks operad:

Theorem 3.6 ([BH15, Thm 4.19])**.** *Let $\mathcal{U}$ be a G-universe, and let $K \leq H \leq G$. Then, $K \leq H$ is admissible in $\underline{\mathcal{C}}(\mathcal{D}(\mathcal{U}))$ if and only if there is an H-equivariant embedding $H/K \to Res^G_H(\mathcal{U})$ (where Res^G_H denotes restriction of representations).*

Proof. Write $H/K = \{h_1K, \dots, h_nK\}$. If H/K is admissible for $\mathcal{D}(\mathcal{U})$, then there exists $x \in \mathcal{D}(\mathcal{U})(n)^{\Gamma_{H,K}}$. The latter being defined as a colimit, we may find a finite-dimensional subrepresentation $V \leq \mathcal{U}$ with $x = (f_1, \dots, f_n) \in \mathcal{D}_V(\mathcal{U})(n)$, where $f_i : \mathbb{D}(V) \to \mathbb{D}(V)$ is an affine embedding for all $i \leq n$. Consider the map $H/K \to \mathcal{U}$, $h_iK \mapsto f_i(0)$. Since the images of the interiors of the disks are disjoint, this map is injective. It is H-equivariant since for any $h \in H$, $h \cdot h_iK = h_{\sigma_h(i)}K$ by definition, and $h \cdot (f_i(0)) = ((h^{-1}, \sigma_{h^{-1}}) \cdot (f_1, \dots f_n))_{\sigma_h(i)}(h^{-1} \cdot 0) = (f_1, \dots, f_n)_{\sigma_h(i)}(0) = f_{\sigma_h(i)}$.

Conversely, if such an embedding $f : H/K \to \mathcal{U}$ exists, we have to find a $\Gamma_{H,K}$-fixed point in $\mathcal{D}(\mathcal{U})(n)$. Up to scaling, we may assume that the image of f is included in the interior of the unit disk in $\mathcal{U}$. Let $\varepsilon > 0$ be a lower bound on the distances between two points in the image of H/K, and the distances between each point and the boundary of the unit disk. Let W be the subrepresentation generated by the image of H/K. It is finite-dimensional because G is finite and H/K too. For each $i \leq n$, consider the disk of radius $\varepsilon/4$ in W around the image of h_iK in $\mathcal{U}$. Let f_i be the affine embedding of $\mathbb{D}(W)$ as this disk (i.e. $\overline{x} \mapsto f(h_iK) + (\varepsilon/4)\overline{x}$). Then $(f_1, \dots, f_n) \in \mathcal{D}_W(\mathcal{U})(n)$, and we want to check that it is a $\Gamma_{H,K}$-fixed point. So we have to prove that for any $h \in H$, $i \leq n$ and $\overline{x} \in W$, we have $h^{-1} \cdot f_{\sigma_h(i)}(h \cdot \overline{x}) = f_i(\overline{x})$. Since h acts by linear isometries, we have

$$\begin{aligned} h^{-1} \cdot ((\varepsilon/4)(h \cdot \overline{x}) + f(h_{\sigma_h(i)}K)) &= h^{-1} \cdot h \cdot ((\varepsilon/4)\overline{x} + h^{-1} \cdot f(h_{\sigma_h(i)}K)) \\ &= (\varepsilon/4)\overline{x} + f(h_iK) = f_i(\overline{x}), \end{aligned}$$

as needed. □

Proposition 3.7 ([Rub21b, Prop. 4.5]). *Let $\mathcal{U} = \bigoplus_{i \in I} V_i$ be a G-universe, where I is an infinite set of indices and V_i is an irreducible real linear isometric representation of G for all $i \in I$. Then $\underline{\mathcal{C}}(\mathcal{D}(\mathcal{U}))$ is generated as a transfer system by $\bigcup_{i \in I} Orb(V_i)$, where for V a G-representation, we define the set of relations:*

$$Orb(V) = \{K \leq G \mid K \neq G \text{ and there is a } G\text{-equivariant embedding } G/K \hookrightarrow V\}.$$

Proof. Let $(\mathcal{T}, \to) = \underline{\mathcal{C}}(\mathcal{D}(\mathcal{U}))$ and let $(\mathcal{T}', \Rightarrow)$ be the transfer system generated by $\bigcup_{i \in I} \mathrm{Orb}(V_i)$. We have to show that $\mathcal{T} = \mathcal{T}'$.

We first show that $\mathcal{T}' \subseteq \mathcal{T}$. Let $(K \leq G) \in \mathrm{Orb}(V_i)$ for some $i \in I$ (in particular $K \Rightarrow G$). Then by definition there is a G-equivariant embedding $G/K \hookrightarrow V_i \hookrightarrow \mathcal{U}$, and thus, by Theorem 3.6, the relation $K \leq G$ is admissible for $\mathcal{D}(\mathcal{U})$, i.e. $K \to G$. Since relations of this form were generators for $\mathcal{T}'$, this proves the first part.

Conversely, let us show that $\mathcal{T} \subseteq \mathcal{T}'$. For this, assume $K \to G$. By Theorem 3.6, there exists a G-equivariant embedding $\varphi : G/K \hookrightarrow \mathcal{U}$. Since G is finite, this factors through $V_{i_1} \oplus \cdots \oplus V_{i_n}$ for some $\{i_1, \dots, i_n\} \subseteq I$. Let $(x_1, \dots, x_n) := \varphi(eK)$.

Then the stabilizer $K = \mathrm{Stab}^G_{x_1,\dots,x_n}$ can be rewritten as $\bigcap_{j\leq n}\mathrm{Stab}^G_{x_j}$. Since by the orbit-stabilizer theorem, $G/\mathrm{Stab}^G_{x_j}$ embeds G-equivariantly into V_{i_j}, we have $\mathrm{Stab}^G_{x_j} \Rightarrow G$ for all $j \leq n$ by construction. Thus, using several times the axioms of closure under restriction and transitivity in the definition of a transfer system, we get $K = \bigcap_{j\leq n}\mathrm{Stab}^G_{x_j} \Rightarrow G$. It remains to show that $K \to H$ implies $K \Rightarrow H$ when $H \neq G$. It suffices to show the cosaturation property (Remark 3.5), i.e. that relations $K \to G$ with target G generate $\underline{\mathcal{C}}(\mathcal{D}(\mathcal{U})) =: \mathcal{T}$. For this, let $\mathcal{T}''$ be the transfer system generated by such relations. Assume $K \to H$ is admissible in $\mathcal{T}$. By Theorem 3.6, there exists an H-equivariant embedding $\varphi : H/K \hookrightarrow \mathrm{Res}^G_H(\mathcal{U})$. Let $x := \varphi(eK)$. Then $K = \mathrm{Stab}^H_x = \mathrm{Stab}^G_x \cap H$. By the orbit-stabilizer theorem, there is a G-equivariant embedding $G/\mathrm{Stab}^G_x \hookrightarrow \mathcal{U}$, so $\mathrm{Stab}^G_x \to G$ by Theorem 3.6. By construction, this relation is also admissible for $\mathcal{T}''$, and thus, by restriction, the relation $\mathrm{Stab}^H_x = \mathrm{Stab}^G_x \cap H \leq H$ is also admissible for $\mathcal{T}''$, as needed. □

Proof of Theorem 3.4.

Step 1: preliminaries from representation theory. Representation theory tells us that an irreducible component of a G-universe can be of two possible kinds. Firstly, it can be one dimensional, in which case each element of G acts as an element of $O(1) \simeq \{\pm 1\} \simeq C_2$. Then, the representation can be written as a map $G \to C_2 \to \mathbb{R}$. Secondly, it can be two-dimensional, in which case each element of G acts as an element of $SO(2)$, i.e. a rotation. Since G is finite, the representation factors as a map $G \to C_{|G|} \to \mathbb{R}^2$ where $C_{|G|}$ is viewed as the subgroup of rotations by multiples of $2\pi/|G|$. Now, if $V \leq \mathcal{U}$ is any irreducible component, with action $\varphi : G \to V$, a simple computation shows that $\mathrm{Orb}(V) = \{\ker(\varphi) \leq G\}$. As we just saw, $G/\ker(\varphi)$ embeds into C_2 or $C_{|G|}$, so this quotient is cyclic.

Step 2: "only if" direction. Assume $\mathcal{T} = \underline{\mathcal{C}}(\mathcal{D}(\mathcal{U}))$ for some G-universe $\mathcal{U}$. Then, by Proposition 3.7, $\mathcal{T}$ is generated by all relations of the form $\ker(\varphi) \leq G$ with $\varphi : G \to V$ an irreducible subrepresentation of $\mathcal{U}$. We just saw that in this situation, $G/\ker(\varphi)$ was cyclic. This proves the first direction of the theorem.

Step 3: "if" direction. Assume that $(\mathcal{T}, \to)$ is a transfer system generated by relations $H_i \to G$ for $i = 1, \dots, n$ with G/H_i cyclic. Thus we may view G/H_i as the cyclic subgroup of $O(2)$ consisting in rotations of angle a multiple of $2\pi/|G/H_i|$. Consider the representation $\varphi_i : G \to G/H_i \to O(2) \to \mathbb{R}^2 =: V_i$. We then have $\mathrm{Orb}(\varphi_i) = \{H_i \leq G\}$ for all $i \leq n$. Therefore, if $\mathcal{U} := \bigoplus_{n\in\mathbb{N}}(\mathbb{R}_{\mathrm{triv}} \oplus V_1 \oplus \dots \oplus V_n)$, by Proposition 3.7 we have $\underline{\mathcal{C}}(\mathcal{D}(\mathcal{U})) = (\mathcal{T}, \to)$, as desired. □

3.2. **Linear isometries operads**

We now turn to a second family of examples of $\mathcal{N}_\infty$-operads.

Definition 3.8. Let $\mathcal{U}$ be a G-universe. The *linear isometries operad* $\mathcal{L}(\mathcal{U})$ associated with $\mathcal{U}$ is the topological operad with $\mathcal{L}(\mathcal{U})(n)$ the space of (non necessarily G-equivariant) isometries $\mathcal{U}^{\oplus n} \to \mathcal{U}$, for all $n \in \mathbb{N}$. The action of S_n and the compositions are defined similarly as for the endomorphism operad (Example 1.2).

Lemma 3.9. *Let $\mathcal{U}$ be a G-universe. The action of G by conjugation as in Definition 3.2 makes $\mathcal{L}(\mathcal{U})$ into an $\mathcal{N}_\infty$-operad. In particular, a G-fixed point in $\mathcal{L}(\mathcal{U})(n)$ is a G-equivariant linear isometry $\mathcal{U}^{\oplus n} \to \mathcal{U}$.*

Remark 3.10. Given $\mathcal{U}$ a G-universe, the linear isometries operad $\mathcal{L}(\mathcal{U})$ and little disks operad $\mathcal{D}(\mathcal{U})$ are in general not equivalent. However, an admissible set for $\mathcal{L}(\mathcal{U})$ is always admissible for $\mathcal{D}(\mathcal{U})$ (this follows from Theorems 3.6 and 3.15).

The question is now to determine which transfer systems correspond to linear isometries operads under the equivalence of Theorem 2.3. Blumberg and Hill found a simple necessary condition called *saturation*.

Definition 3.11. A transfer system $(\mathcal{T}, \to)$ on G is called saturated if for all subgroups $K \leq H \leq G$, if $K \to H$ and $K \leq M \leq H$ then $K \to M$ and $M \to H$.

However, this condition does not suffice in general (Remark 3.18). Rubin conjectured in [Rub21b] that it becomes sufficient for cyclic groups of suitable order:

Conjecture 3.12 (Rubin's saturation conjecture)**.** *Let $k \in \mathbb{N}^+$ and $e_1, \dots, e_k \in \mathbb{N}$. There exist integers $s_1, \dots, s_k$ depending on this choice, such that for all k-uples of distinct primes $p_1, \dots, p_k$ with $p_i \geq s_i$ for all $i \leq k$, and $G := C_{p_1^{e_1} \cdots p_k^{e_k}}$, any saturated transfer system on G is realized by some linear isometries operad.*

Several particular cases were first proven: that of cyclic groups of orders p^n and pq ([Rub21b]), $p^n q$ ([HMOO22]) and $p^n q^m$ ([Ban23]), for $p, q \geq 5$ distinct primes and $n, m \in \mathbb{N}$. Finally, MacBrough solved the conjecture, and proved even more:

Theorem 3.13 ([Mac23, Thm 3.5])**.** *Let G be a finite cyclic group of order coprime to 6. Then G satisfies the saturation conjecture, i.e. every saturated transfer system on G can be realized by a linear isometries operad.*

Theorem 3.14 ([Mac23, Thm 3.14])**.** *There exists some function $f : \mathbb{N} \to \mathbb{N}$ with the following property: for any (finite) abelian group G admitting a presentation with at most two generators, if for all primes p dividing $|G|$, we have $p \geq f(\log_p(|P|))$, where P is the Sylow p-subgroup (i.e. a maximal subgroup of order a power of p), then G satisfies the saturation conjecture.*

In the case of a cyclic group C_n, Rubin reduced the problem to a "pleasant puzzle in modular arithmetic" ([Rub21b, p310]). More precisely, admissible relations for $\mathcal{L}(\mathcal{U})$ can be described in terms of the translation-invariance properties of a certain subset of C_n characterizing the universe $\mathcal{U}$, called *indexing set*. To prove this, we first need an analog of Theorem 3.6 for linear isometries operads.

Theorem 3.15 ([BH15, Thm 4.19]). *Let $\mathcal{U}$ be a G-universe, and let $K \leq H \leq G$. Then, $K \leq H$ is admissible in $\underline{\mathcal{C}}(\mathcal{L}(\mathcal{U}))$ if and only if there exists an H-equivariant embedding $\mathbb{Z}[H/K] \otimes \mathcal{U} \to \mathcal{U}$.*

Remark 3.16. To define the action of H on $\mathbb{Z}[H/K] \otimes \mathcal{U}$, choose an enumeration $H/K = \{h_1K, \ldots, h_kK\}$. For all $h \in H$ and $i \leq k$, write $hh_i = h_{\sigma_h(i)}k_i(h)$ with $k_i(h) \in K$. Then, for all $u \in \mathcal{U}$, let $h \cdot (h_iK) \otimes u = h_{\sigma_h(i)}K \otimes k_i(h)u$. In particular, $\mathbb{Z}[H/K] \otimes \mathcal{U}$ is then isomorphic as an H-representation to $\mathbb{Z}[H] \otimes_{\mathbb{Z}[K]} \mathcal{U}$.

Proof. A fixed point $F \in \mathcal{L}(\mathcal{U})(n)^{\Gamma_{H,K}}$ is a $\Gamma_{H,K}$-equivariant map $F : \mathcal{U}^{\oplus n} \to \mathcal{U}$. By definition $\Gamma_{H,K} = \{(h, \sigma_h) \mid h \in H\}$ where σ_h describes the permutation induced on H/K by h. Then, under the identification $\mathbb{Z}[H/K] \otimes \mathcal{U} \cong \mathcal{U}^{\oplus n}$ sending $h_iK \otimes u$ to h_iu on the i-th summand, F becomes an H-equivariant embedding $f : \mathbb{Z}[H/K] \otimes \mathcal{U} \to \mathcal{U}$. Indeed, for all $h \in H$, $i \leq n$ and $u \in \mathcal{U}$, we have:

$$\begin{aligned} h \cdot f(h_iK \otimes u) &= h \cdot F(0, \ldots, h_iu, \ldots, 0) && (h_iu \text{ in the } i\text{-th summand}) \\ &= h \cdot ((h^{-1}, \sigma_h^{-1}) \cdot F)(0, \ldots, h_iu, \ldots, 0) \\ &= hh^{-1}F(0, \ldots, hh_iu, \ldots, 0) && (hh_iu \text{ in the } \sigma_h(i)\text{-th summand}) \\ &= f(h_{\sigma_h(i)}K \otimes k_i(h)u) = f(h \cdot (h_iK \otimes u)). \end{aligned}$$

The proof of the other implication is similar. □

Proposition 3.17 ([Rub19b, Prop. 5.13 and 5.14]). *Let $G = C_n$ for $n \in \mathbb{N}^+$ be the finite cyclic group of order n. Then:*

1. *Any G-universe is of the form $\mathcal{U}_I := \bigoplus_{n\in\mathbb{N}} \bigoplus_{j\in I} \lambda_n(j)$ with $0 \in I \subseteq C_n$ and $-I \subseteq I$, where $\lambda_n(j)$ is the representation of G where $[1] \in G$ acts by rotation by $2\pi j/n$ in $\mathbb{R}^2$. We call such a set I an* indexing set*, and say that it* realizes *the associated transfer system $\underline{\mathcal{C}}(\mathcal{L}(\mathcal{U}_I))$.*
2. *The relation $C_d \cong (n/d)\mathbb{Z}/n\mathbb{Z} \leq (n/e)\mathbb{Z}/n\mathbb{Z} \cong C_e$ for $d \mid e \mid n$ is admissible in $\underline{\mathcal{C}}(\mathcal{L}(\mathcal{U}_I))$ if and only if $(I + d) \pmod e = I \pmod e$.*

Proof. Part 1 follows from Definition 3.1 using the facts that $\lambda_n(0) \cong \mathbb{R}_{\text{triv}} \oplus \mathbb{R}_{\text{triv}}$; that $\lambda_n(i) \cong \lambda_n(n-i)$ for all $0 < i < n$; and that the isomorphism classes of irreducible finite-dimensional representations of G are $\{\mathbb{R}_{\text{triv}}, \lambda_n(1), \ldots, \lambda_n((n-1)/2\}$ if n is odd, and if n is even, one has to add the sign representation to the list.

For part 2, let I be an indexing set and $\mathcal{U}_I$ be the associated G-universe. In virtue of Theorem 3.15, the relation $(n/d)\mathbb{Z}/n\mathbb{Z} \leq (n/e)\mathbb{Z}/n\mathbb{Z}$ is admissible in $\underline{\mathcal{C}}(\mathcal{L}(\mathcal{U}_I))$ if and only if there is exists a $C_e \cong (n/e)\mathbb{Z}/n\mathbb{Z}$ equivariant embedding

$$\mathbb{Z}\left[((n/e)\mathbb{Z}/n\mathbb{Z})\middle/((n/d)\mathbb{Z}/n\mathbb{Z})\right] \cong \mathbb{Z}\left[d\mathbb{Z}/e\mathbb{Z}\right] \otimes \mathcal{U}_I \longrightarrow \mathcal{U}_I.$$

An explicit computation ([Rub19b, Lemma 5.12]) shows that the left-hand side is isomorphic to $\bigoplus_{n\in\mathbb{N}}\bigoplus_{i\in I}\bigoplus_{a=0}^{e/d-1}\lambda_e(i+da)$ and the right-hand side to $\bigoplus_{n\in\mathbb{N}}\bigoplus_{i\in I}\lambda_e(i)$. If such an embedding exists, then each $\lambda_e(i+da)$ embeds in $\bigoplus_{n\in\mathbb{N}}\bigoplus_{i\in I}\lambda_e(i)$. By irreducibility, this can happen only if $i+da$ or $-(i+da) \pmod e$ appears as an index on the right-hand side. Since $I = -I$ we obtain $i+da \in I$ for all $i \in I$ and $0 \leq a \leq e/d-1$. If $e \neq d$ then $d \mid e$ implies $e/d - 1 \geq 1$. Choosing $a = 1$, we thus obtain $(I+d) \pmod e \subseteq I \pmod e$, and choosing $a = e/d-1$, we get $(I+(e/d-1)d) \pmod e = (I-d) \pmod e \subseteq I \pmod e$, so $(I+d) \pmod e = I \pmod e$. Conversely, if the latter holds, then for all $i+da$, $i \in I$ and $0 \leq a \leq e/d-1$, we can embed all $\lambda_e(i+da)$ into the direct sum on the right-hand side since $i+da \in I$ and our sums are both countably infinite. $\square$

Remark 3.18. Here is a counterexample to Theorem 3.13 if 2 or 3 divides $|G|$. We claim that if $p \leq 3$ and $G := C_{p^n q^m}$ for some $n, m \in \mathbb{N}^+$, then the saturated transfer system $(\mathcal{T}, \to)$ on G with only non-trivial relation $\{0\} \to C_q$ is not realized by any linear isometries operad. Indeed, if $I \subseteq C_{p^n q^m}$ was an indexing set realizing it, then by Proposition 3.17(2), $J := I \pmod{pq}$ realizes the restriction of $\mathcal{T}$ to C_{pq}. This is impossible, as proven in [Rub21b, Lemma 5.22]. By contradiction, we would have $J \subseteq pC_{pq}$, because otherwise $J \pmod p \neq \{0\}$, but then $p \leq 3$ implies that the indexing set $J \pmod p$ is equal to C_p. This is 1-translation-invariant, so $\{0\} \to C_p$, which is false. Now, since $\{0\} \to C_q$, $J \pmod q$ is 1-translation-invariant, so $J \pmod q = C_q$. Thus $J = pC_{pq}$ (if $|J| < q$ then also $|J \pmod q| < q$). But then J is p-translation-invariant, so $C_p \to C_{pq}$, which is a contradiction.

Using Proposition 3.17, one can construct by hand an indexing set realizing a given saturated transfer system in easy cases (this is how the proofs of the aforementioned particular cases of Conjecture 3.12 proceed). The combinatorics quickly become pretty involved, and generalizing this approach seems impossible in practice. The main problem is that (saturated) transfer systems on a product of groups (or posets) are not described by (saturated) transfer systems on the individual factors. Thus, the difficulty increases with the number of prime divisors of the order of G. MacBrough's approach beautifully circumvents the problem by defining the notion of *tight pairs*, which are abstract devices associated with the group itself, and whose existence witnesses that G satisfies the saturation conjecture. In particular, there is an algorithm that, given a tight pair and a saturated transfer system, produces a linear isometries operad realizing the transfer system. The process does not depend on any explicit feature of the specific group or saturated transfer system considered, contrarily to the explicit approaches to Conjecture 3.12 mentioned previously. The most significant advantage of tight pairs is that they behave well with respect to products of groups (Proposition 3.25). Any cyclic group being a product of groups of the form C_{p^n} with p prime and $n \in \mathbb{N}$, it then suffices to prove the saturation conjecture for these simpler groups. We now delve into some details of this approach. For a finite group H, let $\widehat{H}$ be the set of isomorphism classes of finite-dimensional irreducible complex representations of H.

Definition 3.19 ([Mac23, Def. 2.2])**.** A *sub-inductor* for G a finite abelian group is a collection J of maps $J_K^H : \mathcal{P}(\widehat{K}) \to \mathcal{P}(\widehat{H})$ from the subsets of $\widehat{K}$ to those of $\widehat{H}$, indexed by intervals $K \leq H$ in $\mathrm{Sub}(G)$, such that for all $K \leq H \leq M \leq G$:

1. The map J_K^H commutes with unions, inclusions, and complex conjugation.
2. We have $J_K^M = J_H^M \circ J_K^H$.
3. We have $\mathrm{Res}_K^H \circ J_K^H = \mathrm{id}$, where $\mathrm{Res}_K^H(E) = \{W \in \widehat{K} \mid \exists V \in E, W \leq \mathrm{Res}_K^H(V)\}$ for all $E \subseteq \widehat{H}$, and the second Res_K^H denotes the usual restriction.
4. For any $K' \leq H$ and $E \subseteq \widehat{K'}$, we have $\mathrm{Res}_K^H J_{K'}^H(E) \subseteq J_{K\cap K'}^K \mathrm{Res}_{K\cap K'}^{K'}(E)$.
5. We have $\mathbb{C}_{\mathrm{triv}} \in J_K^H(\mathbb{C}_{\mathrm{triv}})$.

Definition 3.20 ([Mac23, Def. 3.1])**.** A *tight pair* for G a finite abelian group is a pair (D,J) where $J = (J_K^H)_{K \leq H \leq G}$ is a sub-inductor; and $D = (D(H))_{H \leq G}$ is a *diagram*, i.e. a collection of subsets $D(H) \subseteq \widehat{H}$ indexed by subgroups of G. We require the following axioms, for all $K \lneq H \leq G$:

1. The diagram D is stable under complex conjugation (i.e. $\overline{D(H)} = D(H)$) and restriction (i.e. $\mathrm{Res}_K^H(D(H)) \subseteq D(K)$).
2. We have $\mathrm{Ind}_K^H(D(K)) \not\subseteq D(H) \cup \bigcup_{H' \lneq H} J_{H'}^H(D(H'))$, where Ind_K^H is extended from the usual induction of representations in the same way we extended Res_K^H.
3. We have $D(H) \not\subseteq \bigcup_{H' \lneq H} J_{H'}^H(D(H'))$.

Remark 3.21. Stability under conjugation of $D(H)$ corresponds to the requirement $I \subseteq -I$ in the definition of an indexing set. If we further require $\mathbb{C}_{\mathrm{triv}} \in D(H)$ for $H \leq G$, then each $D(H)$ corresponds to a G-universe (Proposition 3.17(1)).

Theorem 3.22 ([Mac23, Cor. 3.2])**.** *Let G be a finite abelian group. If G admits a tight pair, then it satisfies the saturation conjecture; that is, every saturated transfer system on G can be realized by a linear isometries operad.*

Remark 3.23. Condition 3 in Definition 3.20 is not needed to prove Theorem 3.22, but ensures well-behavedness of tight pairs under products (Proposition 3.25).

Sketch of proof. Given a tight pair (D, J) for G, we may assume $\mathbb{C}_{\mathrm{triv}} \in D(H)$ for all $H \leq G$ (otherwise, add it to the diagram, the result remains a tight pair). Then, every $D(H)$ is stable under conjugation and contains the trivial representation by hypothesis, so it corresponds to a G-universe by Remark 3.21.

Given a saturated transfer system $(\mathcal{T}, \to)$ on G, we will extend D into an $(\mathcal{T}, \mathrm{Ind})$-stable and $(\leq, J)$-stable diagram $\widetilde{D}$. Here, for a transfer system $\mathcal{T}'$ and a sub-inductor J', a diagram D' is called $(\mathcal{T}', J')$-stable if $(J')^H_K(D'(K)) \subseteq D'(H)$ for all relations $K \leq H$ admissible for $\mathcal{T}'$. Mac Brough proves that such stability properties and the axioms in the definition of a tight pair ensure that the G-universe corresponding to $\widetilde{D}(G)$ realizes $\mathcal{T}$. The $(\mathcal{T}, \mathrm{Ind})$-stability of the diagram will ensure $\mathcal{T} \subseteq \underline{\mathcal{C}}(\widetilde{D}(G))$, while the other assumptions imply the converse.

To find such a diagram $\widetilde{D}$, one repeats the process of alternatively taking the minimal $(\mathcal{T}, \mathrm{Ind})$-stable diagram, respectively $(\leq, J)$-stable diagram containing the diagram from the previous step, starting with D. By finiteness of the objects considered, this process stabilizes after a finite number of steps. In formulas, one computes the sequence $D =: D_0 \subseteq D_1 \subseteq \dots$ where, if $i \geq 1$ is odd, we have

$$\forall H \leq G,\ D_i(H) = \bigcup_{K \to H} \mathrm{Ind}^H_K(D_{i-1}(K))$$

(minimal $(\mathcal{T}, \mathrm{Ind})$-stable extension of D_{i-1}) and if $i \geq 2$ is even, we have

$$\forall H \leq G,\ D_i(H) = \bigcup_{K \leq H} J^H_K(D_{i-1}(K))$$

(minimal $(\leq, J)$-stable extension of D_{i-1}). □

Proposition 3.24 ([Mac23, Lemma 3.4])**.** *Let $p \geq 5$ be a prime and $n \in \mathbb{N}$. Then, the group $G = C_{p^n}$ admits a tight pair.*

Sketch of proof. Subgroups in G form a chain $H_0 \subseteq H_1 \subseteq \cdots \subseteq H_n = G$ with $H_i \cong C_{p^i}$ for all $0 \leq i \leq n$. Define a sub-inductor J for G as follows: for all $i \leq n$, let $J^{H_i}_{H_i}$ be the identity, $J^{H_i}_{H_{i-1}}$ be the union-preserving extension of

$$J^{H_i}_{H_{i-1}}(\{\lambda_{p^{i-1}}(j)\}) := \{\lambda_{p^i}(j), \lambda_{p^i}(p^i - (p^{i-1} - j))\},$$

and $J^{H_{i+k}}_{H_i} = J^{H_{i+k}}_{H_{i+k-1}} \circ \cdots \circ J^{H_{i+1}}_{H_i}$ for any $1 \leq k \leq n - i$. One has to check that this indeed defines a sub-inductor. Also let $D(H_i) = \{\mathbb{C}_{\mathrm{triv}}, \tau_i, \overline{\tau_i}\}$ for all $i \leq n$, where we choose $\tau_i := \lambda_{p^i}(2p^{i-1})$. MacBrough's proof requires $\tau_i \in \mathrm{Ind}^{H_i}_{H_{i-1}}(\mathbb{C}_{\mathrm{triv}}) \backslash \bigcup_{j<i} J^{H_i}_{H_j}(\widehat{H_j})$.

The latter equals

$$\{\lambda_{p^i}(\alpha p^{i-1}) \mid 0 \leq \alpha < p\} \setminus \{\lambda_{p^i}(j), \lambda_{p^i}(p^i - (p^{i-1} - j)) \mid 0 \leq j < p^{i-1}\},$$

and $p > 3$ implies $p^{i-1} < 2p^{i-1} < p^i - p^{i-1}$, so $\tau_i = \lambda_{p^i}(2p^{i-1})$ is a valid choice. Using Frobenius reciprocity and cardinality arguments (where the fact that $p > 3$ comes again in handy), MacBrough shows that (D, J) is a tight pair for G. □

Proposition 3.25 ([Mac23, Lemma 3.3])**.** *Let (D, J) and (D', J') be tight pairs for finite abelian groups G, respectively G', with coprime orders. Then $G \times G'$ admits a tight pair $(D \otimes D', J \otimes J')$, with, for all $K \leq H \leq G$ and $K' \leq H' \leq G'$,*

$$\begin{aligned}(D \otimes D')(H \times H') &= D(H) \otimes D'(H') \\ &= \{Z \in \widehat{H \times H'} \mid \exists V \in D(H), \exists W \in D(H'), Z \leq V \otimes W\}\end{aligned}$$

and $(J \otimes J')^{H \times H'}_{K \times K'}$ is the union-preserving extension of the assignment

$$\forall \tau \in \widehat{K}, \forall \tau' \in \widehat{K'}, \quad (J \otimes J')^{H \times H'}_{K \times K'}(\tau \otimes \tau') = J^H_K(\tau) \otimes J^H_K(\tau).$$

The statement is not too difficult to prove by checking the axioms directly. The core of Theorem 3.13 is rather to define tight pairs, in a way that they witness satisfaction of the saturation conjecture while being compatible with products.

Proof of Theorem 3.13. Let G be a cyclic group of order n coprime to 6. Write the prime decomposition of n as $p_1^{e_1} \cdots p_r^{e_r}$. By assumption, we have $p_i \geq 5$ for all $i \leq r$. Then, by Proposition 3.24, for all $i \leq r$, the group $C_{p_i^{e_i}}$ admits a tight pair. Using Proposition 3.25 $(n-1)$ times, the group $\prod_{i \leq r} C_{p_i^{e_i}} \cong G$ admits a tight pair. Therefore, by Theorem 3.22, G satisfies the saturation conjecture. □

4. **Open problems and further directions**

This section aims to give a starting point to the interested reader wishing to continue the adventure; it contains references, some information about recent developments in the field, and several suggestions of open questions to consider.

First of all, for readers wishing to learn more, [BH15] constitutes a good introduction to $\mathcal{N}_\infty$-operads, and [LV12] and [MSS02] are good references for studying operads in general. About Steiner (or little disks) and linear isometries operads, Rubin's and MacBrough's articles [Rub21b] and [Mac23] are pretty accessible.

As for further developments, the authors of [CGM24] study different variants of the problem of Section 3, namely that of realizing $\mathcal{N}_\infty$-operads by Steiner (or little disks) and linear isometries operads. More precisely, working over *complex* G-universes instead of real ones, they study which pairs of transfer systems correspond to pairs consisting of a little disks operad and a linear isometries operad over the same G-universe. They also study the saturation conjecture for cyclic groups with order not coprime to 6. The situation is not entirely settled, and the authors make a conjecture, which they partly prove ([CGM24, Conjectures 5.1 and 5.2]).

In [Mac23, Questions 1.4 and 1.5], MacBrough suggests two open problems. Firstly, he proposes studying the saturation conjecture for non-abelian groups (for example, determine all integers n such that the dihedral group D_n satisfies the saturation conjecture). In the non-abelian world, interesting progress is made in the article [BMO23]. The main point is that when G is not abelian, transfer systems in $\mathrm{Sub}(G)$ are no longer purely combinatorial objects because one has to take into account the non-trivial conjugation action of G. The axiom of stability under conjugation in the definition of a transfer system could give a hint in the direction of studying the poset of conjugacy classes of subgroups in G instead; however, this fails to describe the homotopy theory of the associated $\mathcal{N}_\infty$-operads in general. The authors provide a criterion on G for this simplification to be legitimate. Secondly, in MacBrough's result for non-cyclic abelian groups (Theorem 3.14 above), an implicit function f appears. One could try to improve the upper bound on f proven in [Mac23], find a lower bound, or discuss the properties of f in general.

Various counting problems have also been addressed. For instance, the authors of [BMO25] provide a formula for the number of homotopy classes of $\mathcal{N}_\infty$-operads on dihedral groups D_{p^n} for p an odd prime and cyclic groups C_{qp^n}, for p, q distinct primes. The authors also provide a "general recursive method for constructing transfer systems on finite lattices", as they write. The authors of [HMOO22] found a closed formula for the number of saturated transfer systems on the group $C_{p^n q^m}$ (and thus, by Theorem 3.13, for the number of homotopy classes of linear isometries operads if $p, q \geq 5$), and for their generating function. Moreover, the authors of [BBR21] study the relation between transfer systems on the group C_{p^n} (and thus, homotopy classes of $\mathcal{N}_\infty$-operads) and various other combinatorial objects such as binary trees, Catalan numbers, and associahedra. Using this, they provide a lower bound on the cardinality of $\mathrm{Ho}(\mathcal{N}_\infty\text{-Op})$ for a given cyclic group.

Transfer systems are also very interesting objects on their own. They are closely related to model structures on (the category associated with) a poset ([FOO$^+$22]), and help in classifying the latter ([BOOR23]). The equivalence between indexing sets and transfer systems has analogs over orbital ∞-categories ([NS22, Ste25]). Moreover, transfer systems are also strongly related to the notion of *bi-incomplete Tambara functors* ([BH22] and [Cha24]) with their norms and transfer maps.

References

[Ban23] Julie E. M. Bannwart. Realization of saturated transfer systems on cyclic groups of order $p^n q^m$ by linear isometries N_∞-operads, 2023. https://arxiv.org/abs/2311.01608.

[BBR21] Scott Balchin, David Barnes, and Constanze Roitzheim. N_∞-operads and associahedra. *Pacific J. Math.*, 315(2):285–304, 2021.

[BH15] Andrew J. Blumberg and Michael A. Hill. Operadic multiplications in equivariant spectra, norms, and transfers. *Adv. Math.*, 285:658–708, 2015.

[BH22] Andrew J. Blumberg and Michael A. Hill. Bi-incomplete Tambara functors. In *Equivariant topology and derived algebra*, volume 474 of *London Math. Soc. Lecture Note Ser.*, pages 276–313. Cambridge Univ. Press, Cambridge, 2022.

[BMO23] Scott Balchin, Ethan MacBrough, and Kyle Ormsby. Lifting N_∞-operads from conjugacy data. *Tunisian Journal of Mathematics*, 5(3):479–504, November 2023.

[BMO25] Scott Balchin, Ethan MacBrough, and Kyle Ormsby. The combinatorics of N_∞ operads for C_{qp^n} and D_{p^n}. *Glasg. Math. J.*, 67(1):50–66, 2025.

[BOOR23] Scott Balchin, Kyle Ormsby, Angélica M. Osorno, and Constanze Roitzheim. Model structures on finite total orders. *Math. Z.*, 304(3):Paper No. 40, 35, 2023.

[BP21] Peter Bonventre and Luís A. Pereira. Genuine equivariant operads. *Adv. Math.*, 381:Paper No. 107502, 133, 2021.

[CGM24] David Cheng, Carl Guo, and Haran Mouli. The realizability problem for linear isometries and steiner operads over cyclic groups, 2024. REU project at the University of Michigan.

[Cha24] David Chan. Bi-incomplete Tambara functors as $\mathcal{O}$-commutative monoids. *Tunis. J. Math.*, 6(1):1–47, 2024.

[FOO+22] Evan E. Franchere, Kyle Ormsby, Angélica M. Osorno, Weihang Qin, and Riley Waugh. Self-duality of the lattice of transfer systems via weak factorization systems. *Homology Homotopy Appl.*, 24(2):115–134, 2022.

[GW18] Javier J. Gutiérrez and David White. Encoding equivariant commutativity via operads. *Algebr. Geom. Topol.*, 18(5):2919–2962, 2018.

[HMOO22] Usman Hafeez, Peter Marcus, Kyle Ormsby, and Angélica M. Osorno. Saturated and linear isometric transfer systems for cyclic groups of order $p^m q^n$. *Topology and its Applications*, 317:108162, 2022.

[Lei11] Tom Leinster. An operadic introduction to entropy. The n-Category Café, May 2011. Accessed March 2025.

[LV12] Jean-Louis Loday and Bruno Vallette. *Algebraic operads*, volume 346 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Heidelberg, 2012.

[Mac23] Ethan MacBrough. Equivariant linear isometries operads over abelian groups, 2023. https://arxiv.org/abs/2311.08797.

[May72] J. Peter May. *The geometry of iterated loop spaces*, volume Vol. 271. Springer-Verlag, Berlin-New York, 1972.

[MSS02] Martin Markl, Steve Shnider, and Jim Stasheff. *Operads in algebra, topology and physics*, volume 96 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2002.

[NS22] Denis Nardin and Jay Shah. Parametrized and equivariant higher algebra, 2022. https://arxiv.org/abs/2203.00072.

[Rub19a] Jonathan Rubin. Categorifying the algebra of indexing systems, 2019. https://arxiv.org/abs/1909.11739.

[Rub19b] Jonathan Rubin. Characterizations of equivariant steiner and linear isometries operads, 2019. https://arxiv.org/abs/1903.08723.

[Rub21a] Jonathan Rubin. Combinatorial N_∞-operads. *Algebraic & Geometric Topology*, 21(7):3513–3568, 2021.

[Rub21b] Jonathan Rubin. Detecting Steiner and linear isometries operads. *Glasg. Math. J.*, 63(2):307–342, 2021.

[Ste25] Natalie Stewart. Equivariant operads, symmetric sequences, and Boardman-Vogt tensor products, 2025. https://arxiv.org/abs/2501.02129.

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