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Beach Math

Problems to do in Ibiza, Spain

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Contents

0. What is Beach Math?

It's 6:00 in the morning, and you're watching the sunrise after a long night of clubbing in Ibiza, Spain. You bite into a warm waffle that you bought from a place nearby that you have already forgotten. As your toes sink into the cool sand and you start to think about the events of the night, your mind starts wandering towards math, towards Beach Math. These problems are for your most contemplative moments, in Ibiza or anywhere you might happen to be. We hope you like them!

1. Problems

1.1. Calm Waves

Problem 1. (From the movie *A Brilliant Young Mind.*) Given a row of cards, some face-up and some face-down, a *move* consists of flipping a face-down card and also the card immediately to its right, if there is one.

Suppose you start with n cards laid out in a row, all face-down, and you keep making moves until you can't anymore. Show that, regardless of the sequence of moves you made, this process must terminate with all cards face-up.

1.2. **Rough Waters**

Problem 2. Alice has a very large house – so large, in fact, that it's unbounded as a subset $E \subseteq \mathbb{R}^3$. Moreover, it's so spacious that her pet bird, Coco, can fly in a straight line between any two points inside the house without any issue; it's convex as a subset $E \subseteq \mathbb{R}^3$. Unfortunately, Coco is a very claustrophobic bird. She needs to be able to fly arbitrarily far in some possible direction from some starting point in the house without running into the walls – otherwise she'll be *unhappy*.

In symbols: if $(X, \|\cdot\|)$ is a normed vector space and $E \subseteq X$, then Coco is *unhappy in* E if and only if $\forall p \in E$, $\forall v \in X \setminus \{0\}$, $\exists t \in [0, \infty)$, $p + tv \notin E$. We'll say Coco is *happy* in E when she is not unhappy in E.

(a) Show that Coco is happy in Alice's house, i.e. show that every unbounded convex subset of $\mathbb{R}^3$ contains an infinite ray.

(Hint: Reach for the moon – try to show for all $n \in \mathbb{N}$ that unbounded convex subsets of $\mathbb{R}^n$ contain an infinite ray. I promise this is easier.)

(b) Alice is leaving for vacation, and has dropped Coco off to stay at Bob's house. Bob was told "make sure your place is convex and unbounded before taking Coco in," and he did so. However, Bob is an infinite-dimensional creature; his house is a convex unbounded subset of the sequence space

$$\ell^\infty = \{x \in \mathbb{R}^{\mathbb{N}} \mid \|x\|_\infty < \infty\},$$

where

$$\|x\|_\infty = \sup_{n \in \mathbb{N}} |x_n|.$$

Must Coco be happy in Bob's house? Explain.

See Figure 1 for a two-dimensional diagram of the situation here.

1.3. **Tsunami**

Problem 3 (Monsky)**.** We will say that an m-tuple of nonnegative real numbers $(a_1, \ldots, a_m)$ is a *dissection of the square* when we may write $[0,1]^2$ as a union of m triangles T_i with disjoint interiors such that a_i is the area of T_i for each i; see Figure 2 for an example. Observe that we must have $\sum_{i=1}^m a_i = 1$ by construction.

(a) Prove that we may dissect the unit square $S = [0,1]^2$ into N equal-area triangles with disjoint interiors, i.e. $(1/N, \ldots, 1/N)$ is a dissection of the square, if and only if N is even.

(b) More generally, if $(a_1, \ldots, a_m)$ is a dissection of the square, show that there is a polynomial p with integer coefficients such that $p(a_1, \ldots, a_m) = \frac{1}{2}$.

Use this to provide another proof that $(1/N, \ldots, 1/N)$ is not a dissection of the square if N is odd.

(*Part (a) is rather challenging, and part (b) even more so.*)

1.4. **Figures for Rough Waters and Tsunami**

Rough Waters:

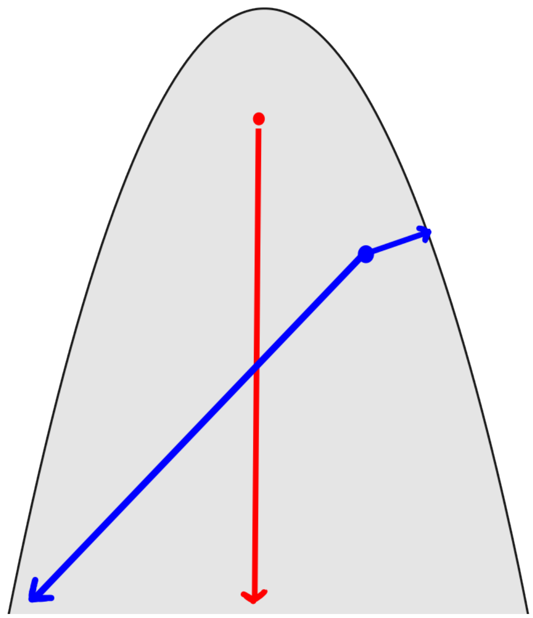

Figure 1: Here's an unbounded convex set $E \subseteq \mathbb{R}^2$, shaded in gray. If Coco lived in E, she would be happy, as she can start at the red point and fly straight in the direction of the red arrow without leaving the set. Note, though, that not every point and every direction works: if Coco starts from the blue point and flies in either of the blue directions, she will eventually leave E. All you need to do is show that some point and some direction works.

Tsunami:

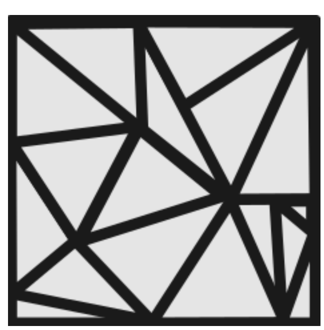

Figure 2: Geometric dissection of the square into $m = 17$ triangles $T_1, \ldots, T_{17}$. The corresponding dissection as we defined it is $(a_1, \ldots, a_{17})$, where T_i has area a_i.

2. **Solutions**

2.1. **Calm Waves**

Let a face-down card have value 1 while a face-up card has value 0. Any state of the game can therefore be written as a sequence of n binary digits, starting at n 1s. Note that any move must necessarily decrease the value of this binary number, and the only way one cannot make a move is if there are no 1s, i.e., all the cards are face-up. Hence, eventually we must reach the state of all 0s.

2.2. **Rough Waters**

Part (a): We'll take the hint and prove that every convex subset of $\mathbb{R}^n$ contains an infinite ray by induction on $n \geq 1$. Convex subsets of $\mathbb{R} = \mathbb{R}^1$ are precisely intervals, and easily every unbounded interval I will contain an interval of the form $(-\infty, a]$ or $[a, \infty)$ for some $a \in \mathbb{R}$. So let $n \in \mathbb{N}$ with $n \geq 1$ and suppose that every unbounded convex subset of $\mathbb{R}^k$ for $1 \leq k \leq n$ contains an infinite ray. Let $C \subseteq \mathbb{R}^{n+1}$ be convex, unbounded. We will assume C has nonempty interior in $\mathbb{R}^{n+1}$; we'll see later how we can discharge this assumption.

Let $p \in C^\circ$, $S^n \subseteq \mathbb{R}^{n+1}$ be the unit sphere, and let's define $f : S^n \to [0, \infty]$ such that $f(v)$ is "the farthest we can travel from p in the direction v while staying inside C." In symbols:

$$f(v) = \sup\{t \geq 0 : p + tv \in C\}.$$

Because p is an interior point of C, f is a continuous function. Towards a contradiction, suppose $f(v) < \infty$ for every $v \in S^n$. Then *by compactness of the unit sphere* (since we're working in a finite-dimensional space), f is bounded, so there is $0 < \alpha < \infty$ such that for every $v \in S^n$, if $p + tv \in C$, then $t \leq \alpha$. (We know $\alpha > 0$ since $p \in C^\circ$ and balls are convex.) If $y \in C$, then the line segment from y to p is in C, and we can write

$$y = p + t\left(\frac{y}{\|y\|}\right) \in C$$

for $t = \|y\|$, and hence $\|y - p\| = t \leq \alpha$ by definition of α. So $C \subseteq B(p, \alpha)$, contradicting the fact that C is unbounded. Hence there is $v \in S^n$ such that $f(v) = \infty$, and hence $p + tv \in C$ for every $t \geq 0$ as desired.

If $C^\circ = \varnothing$, then convexity implies that C is contained in an affine subspace of lower dimension, and hence by the induction hypothesis C contains an infinite ray in said subspace, which is an infinite ray in $\mathbb{R}^{n+1}$. Hence we conclude that every unbounded convex subset of $\mathbb{R}^n$ contains an infinite ray for $n \geq 1$. In particular, we can settle the question raised in part (a): Coco is happy in Alice's house.

Part (b): Unfortunately, Coco need not be happy in Bob's house; we might expect this since the proof for (a) relied on compactness of the unit sphere, which is a fundamentally finite-dimensional phenomenon. (Indeed, compactness of the unit sphere of a normed space is *equivalent* to it being finite-dimensional.) Suppose Bob's house is given by the set

$$C^* = \operatorname{conv}\{ne_n : n \in \mathbb{N}\},$$

where e_n is the sequence with a 1 at position n and zeroes elsewhere, and conv denotes the convex hull, which can be constructed impredicatively: $\operatorname{conv} E$ is the intersection of all convex sets containing E; it's the smallest convex set containing E. Observe that $C^* \subseteq \ell^\infty$ is convex by definition, and unbounded since

$$\|ne_n\|_\infty = n \to \infty$$

as $n \to \infty$.

We claim C^* contains no infinite ray. Let $p \in C^*$, $v \in \ell^\infty$ with $\|v\|_\infty = 1$, $t \geq 0$. If $p + tv \in C^*$, then $|p_n + tv_n| \leq n$ for every n. Choose n such that $v_n \neq 0$; without loss of generality we may assume $v_n > 0$. Then we may always choose t large enough so that $|p_n + tv_n| > n$, and hence $\sup\{t \geq 0 : p + tv \in C^*\} < \infty$ for every $v \in \ell^\infty$ with $\|v\|_\infty = 1$. That is, C^* contains no infinite ray, as desired.

Remark 2.1. The set C^* has empty interior in ℓ^∞. If we assume Bob's house has *nonempty* interior in ℓ^∞, can we guarantee that Coco is happy? Or is there still a chance she is unhappy?

2.3. Tsunami

Part (a): If $N \geq 1$ is even, write $N = 2k$ for $k \in \mathbb{N}$ and break

$$[0,1]^2 = \bigcup_{i=0}^{k-1} \left(\left[\frac{i}{k}, \frac{i+1}{k} \right] \times [0,1] \right)$$

into k vertical rectangles of equal area. Cutting each one along the diagonal witnesses that $(1/N, \dots, 1/N)$ is a dissection of the square. Proving that

$$(1/N, \dots, 1/N) \text{ is a dissection } \Rightarrow N \text{ is even}$$

is *much* harder; we'll need some clever tools from topology and algebra. We follow the excellent exposition of Moragues, 2016, as well as Monsky's original 1970 paper.

The first tool we'll need is the two-dimensional version of *Sperner's lemma*:

Lemma 2.2. *Let Σ be a triangulation of an equilateral triangle Δ, and label its vertices $1, 2, 3$ such that vertices of Δ receive pairwise distinct labels, and any vertex refining the edge from vertex i to vertex j of Δ receives label i or j. Then Σ has an odd number of rainbow triangles: triangles labeled $1, 2, 3$.*

To prove this, we recall the *handshake lemma*: if $G = (V, E)$ is a finite graph, then

$$\sum_{v \in V} \deg(v) = 2|E|.$$

Proof of Sperner's lemma. Declare every triangle of Σ to be the vertex of a graph, and add an additional vertex on the outside. Draw an edge between vertices v, w whenever they are separated by a 1-2 edge. The properties of the coloring ensure that the outside vertex has odd degree. We claim every interior vertex has degree 0, 1, or 2. Indeed:

- a triangle could have all its vertices colored without both 1 and 2; this corresponds to degree 0.
- a triangle could have vertices colored 1 and 2, but not rainbow: this corresponds to degree 2.
- a triangle could be rainbow; this corresponds to degree 1.

By the handshake lemma, every finite graph has an even number of vertices of odd degree, and thus this graph has an odd number of interior vertices with odd degree. But these correspond exactly to rainbow triangles. □

We actually need a slightly different version of Sperner's lemma with a very similar proof, since we're triangulating a square:

Lemma 2.3. *Let P be a polygon whose vertices are colored by three colors $1, 2, 3$, and Σ a triangulation such that vertices refining an edge between i and j receive colors i or j. Then the number of rainbow triangles has the same parity as the number of 1-2 edges on the boundary of the polygon.*

We'll also need to use some algebra. In particular, we recall the 2-adic valuation and corresponding 2-adic absolute value on $\mathbb{Q}$: given $a \in \mathbb{Q}$, write $a = 2^n(r/s)$ uniquely where $n \in \mathbb{Z}$, $\gcd(r, s) = 1$ and r, s have no factors of 2. The 2-adic valuation of a is $v_2(a) = n$ (and $v_2(0) = \infty$) and the corresponding absolute value on $\mathbb{Q}$ is given by

$$|a|_2 = 2^{-n}.$$

$|\cdot|_2$ is a *non-Archimedean* absolute value: $|0|_2 = 0$, $|xy|_2 = |x|_2|y|_2$, and

$$|x + y|_2 \leq \max\{|x|_2, |y|_2\}.$$

For $n \in \mathbb{Z}$, $|n|_2 < 1$ iff n is even; this will be the key to our proof.

There's just one problem with this whole setup. The 2-adic absolute value only makes sense for points in $\mathbb{Q}$, so if we want to, say, apply it to the coordinates of the vertices in a $(1/N, \ldots, 1/N)$-dissection of the square, then *a priori* we'd only be able to do so if our vertices were in $\mathbb{Q}^2$. However, by a bit of algebraic dark magic (i.e. Zorn's lemma, localization, and the theory of valuation rings: *Chevalley's theorem*) we can actually *extend* $|\cdot|_2$ to a non-Archimedean absolute value on $\mathbb{R}$, i.e. (using the same symbol) there is $|\cdot|_2 : \mathbb{R} \to \mathbb{R}$ such that $|0|_2 = 0$, $|xy|_2 = |x|_2|y|_2$,

$$|x + y|_2 \leq \max\{|x|_2, |y|_2\}$$

for all $x, y \in \mathbb{R}$, and $|x|_2$ is the 2-adic absolute value of x for $x \in \mathbb{Q}$. In particular, it's still true for every $n \in \mathbb{Z}$ that

$$n \text{ is even} \iff |n|_2 < 1.$$

Now, we may prove that if $(1/N, \ldots, 1/N)$ is a dissection of the unit square, then N is even. We partition $\mathbb{R}^2 = S_1 \sqcup S_2 \sqcup S_3$, where

$$S_1 = \{(x, y) : |x|_2 < 1,\ |y|_2 < 1\},$$

$$S_2 = \{(x, y) : |x|_2 \geq 1,\ |x|_2 \geq |y|_2\},$$

$$S_3 = \{(x, y) : |y|_2 \geq 1,\ |y|_2 > |x|_2\},$$

and color the points in S_i by i. Moreover, observe that $a + S_2 = S_2$ and $a + S_3 = S_3$

for all $a \in S_1$, by the ultrametric inequality $|x+y| \leq \max\{|x|, |y|\}$.

Proposition 2.4. *Let T be a triangle with one vertex in each S_i and let A denote its area. Then $|A|_2 > 1$.*

Proof. Observe $0 \in S_1$, so by translation-invariance we may assume its S_1-vertex is $(0,0)$. Let $(x_2, y_2) \in S_2$ and $(x_3, y_3) \in S_3$ be its other vertices. Then A is given by the determinant

$$A = \frac{1}{2} \det \begin{pmatrix} x_2 & x_3 \\ y_2 & y_3 \end{pmatrix} = \frac{1}{2}(x_2 y_3 - x_3 y_2).$$

By our coloring, we have $|x_2|_2 \geq |y_2|_2$ and $|y_2|_2 > |x_2|_2$, so that $|x_2 y_3|_2 > |x_3 y_2|_2$. Thus

$$|A|_2 = \left|2^{-1}\right|_2 |x_2 y_3 - x_3 y_2|_2 = 2|x_2 y_3|_2 = 2|x_2|_2 |y_3|_2 \geq 2 > 1$$

as desired. □

Now, suppose $(1/N, \ldots, 1/N)$ is a dissection of the unit square, and let T witness this; T cuts $[0,1]^2$ into N triangles of equal area. Color their vertices of each triangle in T by $i \in \{1, 2, 3\}$ determined by membership in S_i. On the edge $(0,0)$-$(0,1)$, $|x|_2 = 0$ identically, so no points there get color 2. On $(1,0)$-$(1,1)$, $|x|_2 = 1$ identically, so no points there get color 1. On $(0,1)$-$(1,1)$, $|y|_2 = 1$ identically, so no points there get color 1. Hence any 1-2 edges lie on the $(0,0)$-$(1,0)$ edge. Moreover, there we have no vertices of color 3 since $|y|_2 = 0$ identically. As $(0,0) \in S_1$ and $(1,0) \in S_2$, the number of 1-2 edges is given by the number of alternations between 1 and 2. This number must be odd, so there are an odd number of 1-2 edges on the boundary of the unit square.

Thus, by Sperner's lemma, some triangle Δ in the dissection is rainbow, i.e. has a vertex in each S_i. Hence its area A satisfies $|A|_2 > 1$. But $[0,1]^2$ has area $NA = 1$, and thus we must conclude $|N|_2 < 1$ since $|N|_2 |A|_2 = 1$. Since N is an integer, this implies that N is even and completes the proof.

Part (b): We handle the "deduce..." first, since it's *much* easier. Let N odd; to show $(1/N, \ldots, 1/N)$ is not a dissection of the square, we need to show there is no $p \in \mathbb{Z}[x_1, \ldots, x_N]$ with $p(1/N, \ldots, 1/N) = 1/2$. But indeed, there are $b_i \in \mathbb{Z}$, $d \in \mathbb{N}$ with

$$p(1/N, \ldots, 1/N) = \sum_{i=0}^{d} \frac{b_i}{N^i} = \frac{\sum_{i=0}^{d} b_i N^{d-i}}{N^d},$$

which will never be $1/2$ since N is odd.

To prove the polynomial fact, suppose $(a_1, \ldots, a_m)$ is a dissection of the square and let $A = \mathbb{Z}[a_1, \ldots, a_m]$. If $2A = A$, then there is $f \in \mathbb{Z}[x_1, \ldots, x_m]$ with $1 = 2f(a_1, \ldots, a_m)$, and we're done. We claim this is always the case. Indeed, suppose not. Then algebraic dark magic kicks in: 2 is contained in a height-one prime ideal of A; take its integral closure in A; it's a discrete valuation ring R. Using a correspondence between valuation rings and absolute values, we can construct a non-Archimedean absolute value $|\cdot|_a$ on the field of fractions of A, extended to $\mathbb{R}$ by the usual Chevalley business, such that $|a_i|_a \leq 1$ for all $1 \leq i \leq m$ and $|2|_a < 1$. Running the same argument as in part (a), using $|\cdot|_a$ instead of $|\cdot|_2$, we find that $|a_i|_a > 1$, a contradiction. For more detail, see the original paper of Monsky, 1970.

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