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Beach Math

Problems to do in San Diego, USA

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What is Beach Math?

Winter storms are coming and going, but we're on a beach trip in San Diego. I'm sitting on the sand with my partner, watching the sunlight sparkle across the sea. The playful shouts of kids and the low rumble of boats blend softly with the waves. Everything feels peaceful. I am pretty sure this is the kind of moment where a normal person would say something very romantic. My brain thinks about this for a second and decides, with great confidence, the correct answer is MATH. Obviously. Suddenly, three problems pop into my head, all perfectly tailored to this beach. Each one is simple to state but seems hard at first. Once the key idea clicks, the solution collapses into a short chain of steps you can comfortably keep in your head. I love them, and I am optimistically assuming my partner will too. If you were on the beach with us, I suspect you would enjoy them as well. To me, that is Beach Math: small, interesting puzzles that sneak into a holiday and somehow still count as "relaxing."

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1. Problems

1.1. Calm Waves

Problem. While walking along a quiet beach, you discover an old salt-worn leather pouch. Inside are 45 magical pearls of three colors, red, blue, and green, with the following distribution:

$$R = 13, \qquad B = 15, \qquad G = 17.$$

You soon figure out the following magic rule: Whenever you take two pearls of different colors and rub them together, they both turn into the third color. The ocean-blue pearls are your favorite. You wonder: what is the maximum possible number of blue pearls you can end up with using the magic rule?

1.2. **Rough Waters**

Problem. After a long day on the beach, you feel tired and decide to take a relaxing ride around the island. Consider a circular road around the island with n gas stations labeled $1, 2, \ldots, n$ in clockwise order. At station i, your car can refuel g_i units of gas, and it requires c_i units of gas to drive from station i to station $i+1$ (with station $n+1$ identified with station 1). Assume that $g_i, c_i \geq 0$ for all i and

$$\sum_{i=1}^{n} g_i \;\geq\; \sum_{i=1}^{n} c_i.$$

Show that there exists a station k such that if you start at station k with an empty tank, then by repeatedly refueling at each station and driving to the next one, you can travel once around the island back to k without ever running out of gas.
Hint. Further than existence, can you design an algorithm that quickly finds such a starting gas station?

1.3. **Tsunami**

Problem. It is late in the evening and you are helping to tidy up a long row of beach umbrellas. As you walk along the shore, you notice that every umbrella has a different height above the sand. The umbrellas rise and fall messily, but still you observe something more general: whenever you walk past more than n^2 beach umbrellas, you can always find more than n umbrellas so that, reading them in their order along the row, their heights either strictly go up one by one or strictly go down one by one. Can you prove that this pattern always shows up?

2. **Solutions**

2.1. **Calm Waves**

Intuitive idea

With some tries, it is easy to find a sequence of moves that gives 44 blue pearls, but it seems very hard to reach all 45 blue. This suggests we should try to prove that 45 blue pearls are impossible. To do that, we look for a quantity that stays unchanged under every move and then use a contradiction: If the all–blue state had a different invariant value from the start, it can never be reached.

Details

First, it is possible to find a state with 44 blue pearls. From

$$(R, B, G) = (13, 15, 17)$$

perform one move $B + G \to R + R$ to get $(15, 14, 16)$, then apply $R + G \to B + B$ fifteen times. We end at

$$(R, B, G) = (0, 44, 1).$$

So 44 blue pearls is attainable.

Next, let's prove that we cannot have all 45 pearls be blue. Define

$$I := R - G.$$

Check each move:

$$R + B \to G + G \quad \Rightarrow \quad I' = (R - 1) - (G + 2) = I - 3,$$

$$B + G \to R + R \quad \Rightarrow \quad I' = (R + 2) - (G - 1) = I + 3,$$

$$R + G \to B + B \quad \Rightarrow \quad I' = (R - 1) - (G - 1) = I.$$

Notice that in every move $I' \equiv I \pmod 3$, which means $R - G \bmod 3$ is invariant.

At the beginning (before we start rubbing the pearls together),

$$I_{\text{start}} = 13 - 17 = -4 \equiv 2 \pmod 3.$$

If all pearls were blue, we would have $(R, G) = (0, 0)$ and

$$I_{\text{goal}} = 0 - 0 = 0 \equiv 0 \pmod 3.$$

Since the value of $R - G$ modulo 3 is invariant, a state with $R - G \equiv 0 \pmod 3$ is not reachable from a state with $R - G \equiv 2 \pmod 3$. In particular, it is impossible to reach $(R, G) = (0, 0)$, so at least one non–blue pearl must remain. Hence

$$R + G \geq 1 \quad \Longrightarrow \quad B = 45 - (R + G) \leq 44.$$

So the number of blue pearls can never exceed 44.

2.2. Rough Waters

Intuitive idea

What a messy problem! Let's just try something silly and design a simple greedy algorithm to see if it works: Go once around the circular road in clockwise order and see if we can find a good starting point. We keep a running fuel balance, and whenever it goes negative, we declare that whole stretch hopeless and restart at the next station. If this process somehow failed everywhere, then by the time we get back to where we began, all stations would lie inside some "hopeless" negative segment. BINGO! That would force the total net gain to be negative, contradicting $\sum g_i \geq \sum c_i$. Can we rigorously prove that this algorithm works?

Details

Define the net gain at station i by

$$d_i := g_i - c_i \qquad (i = 1, \ldots, n),$$

so d_i is the change in fuel after refueling at i and driving to $i+1$. Our hypothesis gives

$$\sum_{i=1}^{n} d_i = \sum_{i=1}^{n} g_i - \sum_{i=1}^{n} c_i \;\geq\; 0.$$

Below we describe a one-pass algorithm which returns a station index k such that all running sums $\sum_{j=k}^{i} d_j$ along one full lap are never negative. Interpreting these sums as fuel levels shows that we never run out of gas.

Algorithm 1 Find a feasible starting gas station

start $\leftarrow 1$ {starting station}
balance $\leftarrow 0$ {running sum of d_i from *start*}
for $i = 1$ **to** n **do**
    balance $\leftarrow$ *balance* $+ d_i$
    if *balance* < 0 **then**
        start $\leftarrow i + 1$ {discard old candidate; try the next station}
        balance $\leftarrow 0$
    end if
end for
return *start* {this will be the desired station k}

Now, we will prove that this algorithm actually returns the desired station index.

Let

$$P_0 := 0, \qquad P_\ell := \sum_{i=1}^{\ell} d_i \quad (\ell = 1, \dots, n),$$

so that $P_n = \sum_{i=1}^n d_i \geq 0$ and, for any $1 \leq a \leq b \leq n$,

$$\sum_{i=a}^{b} d_i = P_b - P_{a-1}.$$

During the algorithm, suppose that at some iteration i, we reset the candidate starting point from $start = s$ to $i + 1$. Just before this reset, we have

$$balance = \sum_{j=s}^{i} d_j = P_i - P_{s-1} < 0,$$

so $P_i < P_{s-1}$. Thus each reset produces an interval $[s, i]$ with negative total sum, and the prefix sum P_i at the end of that interval is strictly smaller than the prefix sum P_{s-1} at its beginning.

Because the indices i run in increasing order, these negative-sum intervals $[s, i]$ are disjoint and lie in $\{1, \dots, n\}$. If the algorithm were to finish with $start = n + 1$, then these intervals would cover all of $\{1, \dots, n\}$, and we would get

$$P_n = \sum_{i=1}^{n} d_i < 0,$$

contradicting $P_n \geq 0$. Hence the returned value must satisfy $1 \leq start \leq n$. Let k denote this final value of `start`.

We next observe what this means for the prefix sums. At the very beginning, the current candidate is 1 and the smallest prefix sum seen so far is $P_0 = 0$. Whenever we reset from s to $i+1$ we have seen that $P_i < P_{s-1}$, so P_i becomes the new smallest prefix sum encountered up to index i. At the end of the loop, when `start` has its final value k, the corresponding prefix sum P_{k-1} is therefore a global minimum:

$$P_{k-1} \leq P_\ell \quad \text{for all } \ell = 0, 1, \dots, n.$$

Now consider driving once around the circle, starting at station k with an empty tank, in the cyclic order

$$k, k+1, \dots, n, 1, \dots, k-1.$$

We check that the fuel level never becomes negative.

For any index i with $k \leq i \leq n$ we have

$$\sum_{j=k}^{i} d_j = P_i - P_{k-1} \;\geq\; P_{k-1} - P_{k-1} = 0.$$

For an index i with $1 \leq i \leq k-1$, the cumulative gain from k to i around the circle is

$$\sum_{j=k}^{n} d_j + \sum_{j=1}^{i} d_j = (P_n - P_{k-1}) + P_i.$$

Here $P_n \geq 0$ by the global assumption, and $P_i \geq P_{k-1}$ by minimality of P_{k-1}, so

$$(P_n - P_{k-1}) + P_i \;\geq\; 0.$$

Thus every cyclic prefix sum starting at k is nonnegative.

Interpreting these cyclic sums as fuel levels, we see that starting at station k with an empty tank, the fuel level never drops below zero during a complete lap. Therefore k is indeed a feasible starting gas station.

2.3. **Tsunami**

Intuitive idea

At first this feels like magic. Why should having more than n^2 umbrellas force a tidy row of more than n going up or down? The key trick is to stop looking at the whole row at once and instead let each umbrella keep a tiny "scorecard": how long of an increasing chain can end here, and how long of a decreasing chain can end here. So each umbrella gets a pair (I, D).

If we assume there is no increasing or decreasing chain of length more than n, then each of the two scores has to be between 1 and n, which means there are at most n^2 different pairs (I, D) in total.

But once you have more than n^2 umbrellas, the pigeonhole principle says two of them must share the same pair. Hmmm, I wonder if that gives some contradiction.

Details

Let there be m umbrellas in the row, with $m > n^2$. Number them from left to right as

$$U_1, U_2, \ldots, U_m,$$

and let h_i be the height of umbrella U_i. All heights are distinct.

For each position i, we look at subsequences that end at U_i and define:

- $I(i)$: the length of the longest strictly increasing subsequence of heights that ends at U_i,
- $D(i)$: the length of the longest strictly decreasing subsequence of heights that ends at U_i.

So each umbrella U_i gets a pair of positive integers $(I(i), D(i))$.

If there is some i with $I(i) \geq n + 1$, then there is an increasing subsequence of more than n umbrellas (their heights strictly increase as you walk from left to right), and we are done. Similarly, if there is some i with $D(i) \geq n + 1$, then there is a decreasing subsequence of more than n umbrellas, and we are done.

Therefore, to get a contradiction, assume that *no* such subsequence exists. Under this assumption, we must have

$$1 \leq I(i) \leq n \quad \text{and} \quad 1 \leq D(i) \leq n$$

for every i. So each pair $(I(i), D(i))$ lies in the $n \times n$ grid

$$\{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\},$$

which contains exactly n^2 different pairs.

Now we claim that no two different umbrellas can have the same pair. In other words, if $i \neq j$, then

$$(I(i), D(i)) \neq (I(j), D(j)).$$

To see this, suppose $i < j$. Since h_i and h_j are different heights, either $h_i < h_j$ or $h_i > h_j$.

Case 1. $h_i < h_j$. Take the longest increasing subsequence that ends at U_i; it has length $I(i)$. Because h_j is taller than h_i, we can append U_j to this subsequence and get a strictly increasing subsequence that now ends at U_j, of length $I(i) + 1$. Hence

$$I(j) \geq I(i) + 1 > I(i),$$

so $I(j) \neq I(i)$, and thus $(I(i), D(i)) \neq (I(j), D(j))$.

Case 2. $h_i > h_j$. Now take the longest decreasing subsequence that ends at U_i; it has length $D(i)$. Since h_j is shorter than h_i, we can append U_j to this subsequence and obtain a strictly decreasing subsequence ending at U_j of length $D(i) + 1$. Thus

$$D(j) \geq D(i) + 1 > D(i),$$

so $D(j) \neq D(i)$, and again $(I(i), D(i)) \neq (I(j), D(j))$.

In both cases, the pair attached to U_j differs from the pair attached to U_i. Therefore all the pairs

$$(I(1), D(1)), (I(2), D(2)), \ldots, (I(m), D(m))$$

are distinct.

But this is impossible: we have m umbrellas and only n^2 different pairs available in the $n \times n$ grid. Since $m > n^2$, the pigeonhole principle says two umbrellas would have to share the same pair, which contradicts our claim.

Thus our initial assumption was wrong: at least one of the numbers $I(i)$ or $D(i)$ is at least $n+1$, and so among the more than n^2 beach umbrellas there always exists a subsequence of more than n umbrellas whose heights are either strictly increasing or strictly decreasing along the shore.

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