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Khintchine's theorem and related topics in Diophantine approximation

See Hai Ong

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Abstract

An expository overview of Khintchine's theorem and related developments in metric Diophantine approximation is presented. We begin with a study of the classical theorem and the construction of the Duffin–Schaeffer counterexample that demonstrate the necessity of the monotonicity assumption. Drawing on results of Gallagher, higher-dimensional analogues in which the monotonicity hypothesis can be removed are then explored. Furthermore, we survey recent generalised Duffin–Schaeffer-type counterexamples, illustrating that monotonicity on a support of full upper density remains insufficient. Finally, a metric version of the Littlewood conjecture is discussed, focusing on fibration-type results and on settings in which weaker substitutes for the full Duffin–Schaeffer conjecture suffice.

Contents

To fix notation, let $\mathbb{N} := \{1, 2, 3, \cdots\}$ denote the set of natural numbers. Unless otherwise specified, let $\varphi(n)$ denote the Euler totient function.

1. Introduction

In number theory, Diophantine approximation, named after the Greek mathematician Diophantus of Alexandria, concerns the approximation of real numbers by rational numbers. A classical elementary result is *Dirichlet's approximation theorem*, which says that for every irrational α, there exist infinitely many integer pairs (p, q) such that $|\alpha - \frac{p}{q}| < \frac{1}{q^2}$. It is natural to ask whether a similar result holds if, instead of $\frac{1}{q^2}$, one were to consider an arbitrary positive real-valued function ψ. In 1924, Alexander Khintchine proved a classical measure-theoretic result ([Khi24]) — assuming that ψ is monotonically decreasing and $\sum_q \psi(q)$ diverges, then for *almost all* α (with respect to **Lebesgue measure**), there exist infinitely many integer pairs (p, q) such that $|\alpha - \frac{p}{q}| < \frac{\psi(q)}{q}$.

Can the assumption of monotonicity in Khintchine's theorem be removed? As shown by R.J. Duffin and A.C. Schaeffer in 1941 ([DS41]), the answer turns out to be *no*, through a clever counterexample. Duffin and Schaeffer conjectured that if the monotonicity hypothesis is removed, then one would have to compensate with a factor of $\varphi(q)$ — for almost every α, the inequality $|\alpha - \frac{p}{q}| < \frac{\psi(q)}{q}$ has infinitely many coprime integer solutions (p, q) *if and only if* the modified sum $\sum_{q \geq 1} \varphi(q) \frac{\psi(q)}{q} = \infty$. This conjecture gave birth to the field of mathematics that is now commonly known as *metric Diophantine approximation*, where "metric" refers to the Lebesgue measure. As a testament to its difficulty, the Duffin–Schaeffer conjecture was only solved recently in 2019 by Dimitris Koukoulopoulos and James Maynard, forming part of the work recognised by Maynard's Fields Medal.

Many other variants of Khintchine's theorem have been extensively studied; refer to [Hau25] for an overview. To briefly list some of these developments —

1. **Higher-dimensional analogues of Khintchine's theorem** — in $\mathbb{R}^d$ for $d \geq 2$, it turns out that the assumption of monotonicity on ψ can be removed without a compensating totient factor $\varphi(q)$.

2. Results in which the Lebesgue measure is replaced by another measure, such as the Hausdorff measure.

3. **Khintchine's Theorem on manifolds**.

4. **Inhomogeneous Khintchine's Theorem** — consider $|\alpha - \frac{p+\gamma}{q}| \leq \frac{\psi(q)}{q}, \gamma \in \mathbb{R}\backslash\{0\}$. The study of inhomogeneous parameters is a recurring theme; for example such parameters appear in metric formulations of the *Littlewood conjecture*.

The article is organised as follows. First, we explain the classical Khintchine's theorem and the Duffin–Schaeffer counterexample demonstrating the need for monotonicity. Then, we move on to analyse higher-dimensional analogues of Khintchine's theorem, in which monotonicity may be removed. Next, we discuss a 2025 paper on general Duffin–Schaeffer-type counterexamples, in which the authors show that even monotonicity on a support of full upper density remains insufficient. Finally, we consider a fibration-type result related to the Littlewood conjecture.

2. The classical Khintchine's theorem

Theorem 1 (Khintchine's Theorem)**.** *Let $\psi : \mathbb{N} \to [0, \infty)$ be monotonically decreasing (i.e. $\psi(n+1) \leq \psi(n)$ for all $n \in \mathbb{N}$), and define*

$$A(\psi) := \left\{ \alpha \in [0,1) : \left| \alpha - \frac{p}{q} \right| < \frac{\psi(q)}{q} \quad \textit{for infinitely many } (p,q) \in \mathbb{Z} \times \mathbb{N} \right\}.$$

Then,

$$\lambda(A(\psi)) = \begin{cases} 0 & \textit{if } \sum_q \psi(q) < \infty \\ 1 & \textit{if } \sum_q \psi(q) = \infty. \end{cases}$$

Here, λ denotes the Lebesgue measure on $\mathbb{R}$.

Remark 2. It makes sense to consider $\alpha \in [0,1)$ rather than the whole of $\mathbb{R}$ since the integer part of a real number does not affect its Diophantine properties. In particular, $|\alpha - \frac{p}{q}| < \frac{\psi(q)}{q} \iff |q\alpha - p| < \psi(q)$. Write $\|q\alpha\| := \min_{k \in \mathbb{Z}} |q\alpha - k|$. Thus, requiring the inequality to hold for infinitely many pairs (p, q) is equivalent to asking that $\|q\alpha\| < \psi(q)$ for infinitely many q. On a generic interval $[k, k+1)$, we have $\|q(\alpha + k)\| = \|q\alpha + qk\|$, where qk is an integer — adding an integer to $q\alpha$ does not change its distance to the nearest integer. Thus, the condition $\|q\alpha\| < \psi(q)$ is invariant under integer translations of α.

For example, *Dirichlet's approximation theorem* states that for every real number $\alpha \in [0,1)$, there are infinitely many rational numbers $\frac{p}{q}$ such that $|\alpha - \frac{p}{q}| \leq \frac{1}{q^2}$. Here, take $\psi(q) = \frac{1}{q}$ in Khintchine's theorem, so that Dirichlet's theorem implies that $\lambda(A(\psi)) = 1$. Now, $\sum_q \frac{1}{q} = \infty$, so that Khintchine's theorem implies $\lambda(A(\psi)) = 1$, agreeing with Dirichlet's theorem.

That $\sum_q \psi(q) < \infty$ implies that $\lambda(A(\psi)) = 0$ in Khintchine's theorem is straightforward. Regard $[0, 1)$ as a probability space equipped with Lebesgue measure, and define the events $E_q = \bigcup_{p\in\mathbb{Z}} \left\{\alpha \in [0, 1) : |\alpha - \frac{p}{q}| < \frac{\psi(q)}{q}\right\}$.

Now,

$$\begin{aligned}
\sum_q \mathbb{P}(E_q) &= \sum_q \mathbb{P}\left(\bigcup_{p\in\mathbb{Z}} \left\{\alpha \in [0, 1) : \left|\alpha - \frac{p}{q}\right| < \frac{\psi(q)}{q}\right\}\right) \\
&\leq \sum_q \sum_{p\in\mathbb{Z}} \mathbb{P}\left(\left\{\alpha \in [0, 1) : \left|\alpha - \frac{p}{q}\right| < \frac{\psi(q)}{q}\right\}\right) \\
&= \sum_q \sum_{p\in\mathbb{Z}} \lambda\left(-\frac{\psi(q)}{q} + \frac{p}{q}, \frac{\psi(q)}{q} + \frac{p}{q}\right) \\
&= \sum_q \sum_{p=\lceil -\psi(q)\rceil}^{\lceil q+\psi(q)-1\rceil} 2\frac{\psi(q)}{q} \\
&\leq \sum_q 2\frac{\psi(q)}{q}(q + 2\psi(q) + 1) \\
&= 2\sum_q \psi(q) + 4\sum_q \frac{\psi(q)^2}{q} + 2\sum_q \frac{\psi(q)}{q} \\
&< \infty.
\end{aligned}$$

The *First Borel–Cantelli lemma* therefore gives $\mathbb{P}(\limsup_{q\to\infty} E_q) = 0$, which is equivalent to the claim that $A(\psi)$ has measure zero.

Assuming $\sum_q \psi(q) = \infty$, however, it is much harder to conclude that $\lambda(A(\psi)) = 1$ since we do not have independence of the events E_q, which would allow the *Second Borel–Cantelli lemma* to be applied. Actually, Khintchine's original paper proved the theorem under the assumption that $q\psi(q)$ is a decreasing function of q. It was Duffin and Schaeffer, who, in their seminal 1941 paper ([DS41]), proved that this assumption could be replaced by the more general condition that $\frac{\psi(q)}{q^c}$ is a decreasing function of q, where c is any real constant.

In the same paper, Duffin and Schaeffer also showed that the monotonicity assumption on $\psi(x)$ in Theorem 1 cannot be dropped. To explain the counterexample that they constructed, it is notationally convenient to regard ψ as a sequence $\{\alpha_q\}$ of non-negative numbers. We need the following two results from elementary number theory —

Lemma 3 (Sum of Divisors Formula)**.** *For any positive integer* $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, *we* *have* $\sigma(n) := \sum_{d|n} d = \prod_{j=1}^{k} \frac{p_j^{\alpha_j+1}-1}{p_j-1}$.

Lemma 4. $\prod_{p \text{ prime}} \frac{p+1}{p}$ *diverges.*

Proposition 5. *Let R and ϵ be positive numbers. (When constructing our counterexample, we will take R large and ϵ small.) Then, there exists a finite sequence $\{\alpha_q\}$ of non-negative numbers such that:*

$$\sum \alpha_q > \frac{1}{2}, \quad \alpha_q = 0 \text{ when } q \leq R,$$

but for $x \in [0,1)$ the inequality $|x - \frac{p}{q}| < \frac{\alpha_q}{q}$ is only satisfied in a set of measure less than ϵ.

Note that we only ask for the inequality to be satisfied *for some* values of p and q. Thus, if $E := \bigcup_{q\in S} \bigcup_{p\in\mathbb{Z}} \left\{x \in [0,1) : |x - \frac{p}{q}| < \frac{\alpha_q}{q}\right\}$, Proposition 5 essentially claims that $\lambda(E) < \epsilon$.

Proof. Without loss of generality, take $\epsilon \leq 1$ since $[0,1)$ has Lebesgue measure 1. Choose $0 < \alpha < \frac{\epsilon}{2}$. Define $N = p_1 \cdots p_k$ to be a product of k distinct primes, each greater than R. Here, k is chosen to be sufficiently large so that the product $\prod_{i=1}^{k}(1 + \frac{1}{p_i})$ is greater than $1 + \frac{1}{2\alpha}$ (Such a k always exists by Lemma 4). Now, define the sequence:

$$\alpha_q = \begin{cases} \frac{q\alpha}{N} & \text{if } q > 1 \text{ and } q \mid N, \\ 0 & \text{otherwise.} \end{cases}$$

By construction, $\alpha_q = 0$ whenever $q \leq R$ since each prime factor in N is greater than R. Let $E_q \subseteq [0,1)$ consist of $q-1$ open intervals of length $\frac{2\alpha_q}{q}$, with centres at $\frac{p}{q}$ (with $p = 1,2,3,\cdots,q-1$), and the two open intervals at $(0, \frac{\alpha_q}{q})$ and $(1 - \frac{\alpha_q}{q}, 1)$. Now, for $q > 1$ and $q \mid N$, *the open intervals consisting of E_q are disjoint*, so that $|E_q| = 2\alpha_q = 2\frac{q\alpha}{N}$. To see why this is true, it suffices to prove that the consecutive intervals $(\frac{p}{q} - \frac{\alpha}{N}, \frac{p}{q} + \frac{\alpha}{N})$ and $(\frac{p+1}{q} - \frac{\alpha}{N}, \frac{p+1}{q} + \frac{\alpha}{N})$ are disjoint; that is:

$$\begin{aligned} \frac{p+1}{q} - \frac{\alpha}{N} &\geq \frac{p}{q} + \frac{\alpha}{N} \\ \iff \frac{1}{q} &\geq \frac{2\alpha}{N} \\ \iff q &\leq \frac{N}{2\alpha}. \end{aligned}$$

But by assumption, $q \leq N \leq \frac{N}{2\alpha}$ since $\epsilon < 1 \implies 2\alpha < 1$.

Now, define $E := \bigcup_q E_q$. We claim that *the sets E_q are included in E_N*, so that $E = E_N$. (Intuitively, E_N has more intervals than each E_q, and each interval in E_N is at least as wide as each interval in E_q.) Indeed, for some open interval $(\frac{p}{q} - \frac{\alpha}{N}, \frac{p}{q} + \frac{\alpha}{N})$ in some E_q, since $q \mid N$, there exists some positive integer q' such that $qq' = N$ and $\frac{p}{q} = \frac{pq'}{qq'} = \frac{pq'}{N}$. Thus, consider the open interval $(\frac{pq'}{N} - \frac{\alpha}{N}, \frac{pq'}{N} + \frac{\alpha}{N}) = (\frac{p}{q} - \frac{\alpha}{N}, \frac{p}{q} + \frac{\alpha}{N})$ in E_N.

Observe that the desired inequality is satisfied for some $x \in [0, 1)$ *if and only if* $x \in E$. It remains to prove that $\sum |E_q| > 1$ and $|E| = |E_N| < \epsilon$. The second inequality is clear. By applying Lemma 3 to N, we have:

$$\sum |E_q| = \frac{2\alpha}{N} \sum_{q|N,\ q>1} q = \frac{2\alpha}{N} \left(\prod_{j=1}^{k} \left(\frac{p_j^2 - 1}{p_j - 1} \right) - 1 \right) = \frac{2\alpha}{N} \left(\prod_{j=1}^{k} (p_j + 1) - 1 \right).$$

Finally, we claim that this expression is greater than $2\alpha \left[\prod \left(1 + \frac{1}{p_i} \right) - 1 \right]$, which is in turn greater than $2\alpha \left(1 + \frac{1}{2\alpha} - 1 \right) = 1$. Thus, this establishes $\sum |E_q| > 1$, which is a contradiction. Indeed, it suffices to show that:

$$\begin{aligned} \frac{1}{N} \left[\prod (1 + p_i) - 1 \right] > \prod \left(1 + \frac{1}{p_i} \right) - 1 \iff \prod (1 + p_i) - 1 &> N \prod \left(1 + \frac{1}{p_i} \right) - N \\ &= \prod (1 + p_i) - N. \end{aligned}$$

Note that the last inequality holds since $N > 1$. □

Theorem 6 (Failure of Khintchine's theorem without Monotonicity)**.** *There exists a sequence $\{\alpha_q\}$ of non-negative numbers such that $\sum_q \alpha_q = \infty$, but the set of points for which $|x - \frac{p}{q}| < \frac{\alpha_q}{q}$ is satisfied (for infinitely many integers p and q) has measure zero.*

Proof. Roughly speaking, we will apply Proposition 5 repeatedly — construct infinitely many finite sequences satisfying the hypotheses of Proposition 5, then patch all of them together to form our desired sequence.

Take a finite sequence $\{\alpha_q^{(1)}\}$ satisfying Proposition 5 with $R_1 = 1$ and $\epsilon = 2^{-1}$. Since this is a finite sequence, there exists R_2 such that $\alpha_q^{(1)} = 0$ when $q \geq R_2$. Thus, let $\{\alpha_q^{(2)}\}$ satisfy Proposition 5 with $R = R_2$ and $\epsilon = 2^{-2}$. Continuing inductively, we can define the infinite sequence $\{\alpha_q\} = \{\alpha_q^{(1)}\} + \{\alpha_q^{(2)}\} + \cdots$ of non-negative numbers satisfying $\sum_{q=1}^{\infty} \alpha_q = \infty$, since each finite sequence component has a sum greater than $\frac{1}{2}$.

It remains to prove that this sequence works. For each $j \geq 1$, define

$$F_j := \bigcup_{R_j < q \leq R_{j+1}} \bigcup_{a \in \mathbb{Z}} \left\{ x \in [0, 1) : \left| x - \frac{a}{q} \right| < \frac{\alpha_q}{q} \right\}.$$

By the construction of the j-th block, $\lambda(F_j) < 2^{-j}$. Let

$$E := \left\{ x \in [0,1) : \left| x - \frac{a}{q} \right| < \frac{\alpha_q}{q} \text{ for infinitely many } (a,q) \in \mathbb{Z} \times \mathbb{N} \right\}.$$

Since $E \subseteq \bigcup_{j=k}^{\infty} F_j$,

$$\lambda(E) \leq \sum_{j=k}^{\infty} \lambda(F_j) < \sum_{j=k}^{\infty} 2^{-j} = 2^{-k+1}.$$

Letting $k \to \infty$, we obtain $\lambda(E) = 0$, as desired. □

Khintchine's theorem gives many avenues of exploration. For example, we can seek for the necessary and sufficient condition(s) on the function $\psi(n)$, so that for almost every $x \in [0,1)$ there exist infinitely many p, q satisfying $|x - \frac{p}{q}| < \frac{\psi(q)}{q}$. As argued above, dropping the condition of monotonicity and simply asking for the divergence of $\sum \psi(q)$ is too naive. Duffin and Schaeffer famously conjectured that, after restricting to reduced fractions, the appropriate necessary and sufficient condition is $\sum \frac{\psi(q)\varphi(q)}{q} = \infty$. This statement became known as the *Duffin–Schaeffer conjecture* and remained unsolved for many years, until it was finally proved in 2019 ([KM20]).

In the next section, we shall study the extension of Khintchine's theorem to simultaneous Diophantine approximations, which can be understood as a higher dimensional analogue of Khintchine's theorem.

3. Khintchine's theorem in higher dimensions

In higher dimensions, we can drop the assumption of monotonicity in Khintchine's theorem. The underlying idea is that reduced fractions for different denominators q have less overlap (in the sense of Theorem 14 below), so that a sufficient number of such points force their union to have large measure, yielding the divergence conclusion.

Fix $r \geq 2$, and write $\mathbf{l} = (l_1, \cdots, l_r)$ be a lattice vector; that is, each component $l_i \in \mathbb{Z}$. For any positive integer n, denote $(\mathbf{l}, n) = \gcd(l_1, \cdots, l_r, n)$. For any $a \geq 0$, let $U(a) = \{y \in \mathbb{R}^r : 0 \leq y_i < a \ (i = 1, 2, \cdots, r)\}$. The main goal of this section is to understand the following theorem, first stated and proved by P.X. Gallagher in [Gal65].

Theorem 7 (Relaxation of monotonicity). *Let $r \geq 2$. For every sequence $\{a_n\}$ between 0 and 1, there are infinitely many solutions $n, \mathbf{l}$ of $n\mathbf{x} - \mathbf{l} \in U(a_n), (\mathbf{l}, n) = 1$, for almost no $\mathbf{x}$ or almost every $\mathbf{x}$, corresponding respectively to whether $\sum a_n^r$ converges or diverges.*

Note that we do not need $\{a_n\}$ to be monotonic; rather, we suppose that it is bounded between 0 and 1. A related result is the higher-dimensional Duffin–

Schaeffer conjecture, proved by Pollington and Vaughan in 1990 [PV90]; this concerns approximation by primitive rational vectors and is distinct from the monotonicity-free Khintchine theorem considered here.

Let $T_N(\mathbf{x})$ be the number of solutions $n, \mathbf{l}$ of $n\mathbf{x} - \mathbf{l}$ with $n \leq N$, and define $S_N := \sum_{n \leq N} a_n^r$. Now, it suffices to consider $\mathbf{x} \in U(1)$, and prove that we either have a set of Lebesgue measure 0 or 1, depending on whether $\sum a_n^r$ converges or diverges. Define the sets

$$A_n := \{\mathbf{x} \in U(1) : n\mathbf{x} - \mathbf{l} \in U(a_n) \text{ (for some } \mathbf{l})\}$$
$$A_n' := \{\mathbf{x} \in U(1) : n\mathbf{x} - \mathbf{l} \in U(a_n), (\mathbf{l}, n) = 1 \text{ (for some } \mathbf{l})\}.$$

Proposition 8. *If $n\mathbf{x} - \mathbf{l} \in U(a_n)$ for some $\mathbf{l}$ with $n \leq N$, then*

1. *$\mathbf{l}$ is determined by n,*
2. *$\mathbf{x}$, $\mathbf{l} \in U(n)$,*
3. *$|A_n| = a_n^r$.*

Proof. For each i, $0 \leq nx_i - l_i < a_n$. This inequality implies that $l_i \leq nx_i < n$. Also, $nx_i - l_i < a_n < 1 \implies l_i > nx_i - 1 \geq -1$. Since $l_i \in \mathbb{Z}$, it follows that $l_i \geq 0$. Thus, $0 \leq l_i < n$ for all i implies that $\mathbf{l} \in U(n)$.

Now, $0 \leq nx_i < a_n \implies l_i \leq nx_i < a_n + l_i \implies \frac{l_i}{n} \leq x_i < \frac{a_n + l_i}{n}$. Thus, x_i lies in an interval of length $\frac{a_n}{n}$. Hence, for a fixed $\mathbf{l}$, $\mathbf{x}$ is contained in a cube of volume $\left(\frac{a_n}{n}\right)^r$. Since $\mathbf{l}$ is a lattice vector in $U(n)$, there are n choices for each component l_i, giving a total of n^r choices of $\mathbf{l}$. Thus, $|A_n| = n^r \cdot \frac{a_n^r}{n^r} = a_n^r$. $\square$

Corollary 9. *The set of $\mathbf{x} \in U(1)$ for which $n\mathbf{x} - \mathbf{l} \in U(a_n), n \leq N$ has at least K solutions $n, \mathbf{l}$, has measure at most $K^{-1} \cdot S_N$.*

Proof. Define the counting function $v(\mathbf{x}) = \sum_{n=1}^{N} 1_{A_n}(x)$, where $1_{A_n}(\mathbf{x})$ is the indicator function of A_n. Clearly, this counts the number of solutions of a specific $\mathbf{x}$, up to a limit N. Define $E_K := \{\mathbf{x} \in U(1) : v(\mathbf{x}) \geq K\}$. We have:

$$\int_{U(1)} v(\mathbf{x}) = \int_{U(1)} \sum_{n=1}^{N} 1_{A_n}(x) = \sum_{n=1}^{N} \int_{U(1)} 1_{A_n}(x) = \sum_{n=1}^{N} |A_n| = S_N,$$

where the last equality follows from Proposition 8. On the other hand, we have:

$$S_N = \int_{U(1)} v(\mathbf{x}) = \int_{E_K} v(\mathbf{x}) + \int_{U(1) \setminus E_K} v(\mathbf{x}) \geq \int_{E_K} v(\mathbf{x}) \geq K|E_K|.$$

Thus, $|E_K| \leq \frac{1}{K} S_N$. $\square$

The convergence half of Theorem 7 now follows immediately. If $\sum a_n^r$ converges, $K^{-1} \cdot S_N \xrightarrow{K\to\infty} 0$, which clearly implies that $n\mathbf{x} - \mathbf{l} \in U(a_n)$ will have infinitely many solutions $n, \mathbf{l}$ for almost no $\mathbf{x}$ (the condition of coprimality in Theorem 7 does not affect this conclusion). To prove the other half of Theorem 7, we need the following tools.

3.1. Estimates of overlap

We seek upper bounds for $|A'_m \cap A'_n|$ and $|A_m \cap A_n|$. Define

$$S'(a,b) := \sum_{\mathbf{j}\neq 0} |U(a) \cap (U(b) + \mathbf{j})|$$
$$S(a,b) := \sum_{\mathbf{j}} |U(a) \cap (U(b) + \mathbf{j})|.$$

Both sums range over lattice vectors $\mathbf{j} \in \mathbb{Z}^r$. Let $\#(V)$ denote the number of lattice points in the set V.

Proposition 10. $S(a,b) = \displaystyle\int_{U(b)} \#(U(a) - \mathbf{x})\ \mathrm{d}\mathbf{x}$.

Proof. Observe that $\#(U(a) - \mathbf{x}) = \sum_{\mathbf{j}\in\mathbb{Z}^r} 1_{U(a)}(\mathbf{x} + \mathbf{j})$. Thus,

$$\begin{aligned}
\int_{U(b)} \#(U(a) - \mathbf{x}) &= \int_{\mathbf{x}\in U(b)} \sum_{\mathbf{j}\in\mathbb{Z}^r} 1_{U(a)}(\mathbf{x} + \mathbf{j}) \\
&= \sum_{\mathbf{j}\in\mathbb{Z}^r} \int_{\mathbf{x}\in U(b)} 1_{U(a)}(\mathbf{x} + \mathbf{j}) \quad \text{(Fubini-Tonelli)} \\
&= \sum_{\mathbf{j}\in\mathbb{Z}^r} |U(b) \cap (U(a) - \mathbf{j})| \\
&= \sum_{\mathbf{j}\in\mathbb{Z}^r} |(U(b) + \mathbf{j}) \cap (U(a) - \mathbf{j} + \mathbf{j})| \quad \text{(Translation Invariance)} \\
&= \sum_{\mathbf{j}\in\mathbb{Z}^r} |U(a) \cap (U(b) + \mathbf{j})| \\
&= S(a,b).
\end{aligned}$$

□

Proposition 11 (Lattice Counting Asymptotic Formula)**.** $\#(U(a) - \mathbf{x}) = a^r + O(a^{r-1}),\ a \geq \frac{1}{2}$.

Proof. Note that $\mathbf{k} \in U(a) - \mathbf{x}$ *if and only if* for all $i = 1, 2, \cdots, r$, we have $-x_i \leq k_i < a - x_i$; this is an interval of length a and thus contains $\lfloor a \rfloor$ or $\lceil a \rceil$ many integers. Thus, for each i, the number of integers in the interval can be expressed as $a+\delta_i, |\delta_i| \leq 1$. This gives:

$$\#(U(a)-\mathbf{x}) = \prod_{i=1}^{r}(a+\delta_i) = a^r + a^{r-1}\left(\sum \delta_i\right) + \cdots$$

Since $a \geq \frac{1}{2}$, $a^{r-2} = \frac{1}{a}a^{r-1} \leq 2a^{r-1}$, $a^{r-3} = \frac{1}{a^2}a^{r-1} \leq 4a^{r-1}$ and likewise for smaller powers of a. Thus, we have $a^{r-1}\left(\sum \delta_i\right) + \cdots = O(a^{r-1})$. □

Lemma 12.

$$S'(a,b) = O(a^r b^r).$$

Proof. Noting that both S' and S are symmetric in terms of a and b, without loss of generality, let $a \geq b$. First assume that $a \geq \frac{1}{2}$. By Propositions 10 and 11, we have

$$\begin{aligned} S(a,b) &= \int_{U(b)} \#(U(a)-\mathbf{x}) \\ &= \int_{U(b)} a^r + O(a^{r-1}) \\ &= a^r b^r + O(a^{r-1})b^r = a^r b^r + a^r b^r O(a^{-1}) \\ &= a^r b^r(1+O(a^{-1})). \end{aligned}$$

Since $a \geq b$, we also have $S(a,b) = S'(a,b) + |U(a) \cap U(b)| = S'(a,b) + b^r$. Now, we have:

$$\begin{aligned} S'(a,b) &= S(a,b) - b^r \\ &= a^r b^r(1+O(a^{-1})) - b^r \\ &= a^r b^r + O(a^{r-1}b^r) - b^r \\ &= a^r b^r(1+O(a^{-1})) \\ &= O(a^r b^r). \end{aligned}$$

To justify why the $-b^r$ term can be absorbed, note that $a \geq \frac{1}{2} \implies 1 \leq 2a \implies 1 \leq 2^{r-1}a^{r-1} \implies b^r \leq 2^{r-1}(a^{r-1}b^r)$, so that $O(a^{r-1}b^r) - b^r = O(a^{r-1}b^r)$. As for $a < \frac{1}{2}$, it is easy to see that $S'(a,b) = 0$, so that the asymptotic bound clearly holds. □

Lemma 13 (Shift Formula of Sets)**.** *Let $A, B > 0$ and let X, Y be vectors. Then,*

$$(U(A)+X) \cap (U(B)+Y) = [U(A) \cap (U(B)+(Y-X))] + X.$$

Proof. This is clear by element chasing. □

Theorem 14. *For integers $m \neq n$,*

$$|A'_m \cap A'_n| = O(|A'_m||A'_n|).$$

Proof. Recall the Möbius identity $\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases}$.

Note that

$$\begin{aligned}
|A'_n| &= \left(\frac{a_n}{n}\right)^r \sum_{\mathbf{l} \in U(n)} \sum_{d|(\mathbf{l},n)} \mu(d) \\
&= \left(\frac{a_n}{n}\right)^r \sum_{d|n} \sum_{\mathbf{l} \in U(n), d|\mathbf{l}} \mu(d) \\
&= \left(\frac{a_n}{n}\right)^r \sum_{d|n} \left(\frac{n}{d}\right)^r \mu(d) \\
&= a_n^r \sum_{d|n} \left(\frac{1}{d}\right)^r \mu(d) \\
&= a_n^r \prod_{p|n} \left(1 - \frac{1}{p^r}\right).
\end{aligned}$$

For $r \geq 2$, note that

$$\frac{1}{\prod_p \left(1 - \frac{1}{p^r}\right)} = \prod_p \left(\sum \frac{1}{p^{kr}}\right) = \sum_n \frac{1}{n^r} = \zeta(r) \implies \prod_{p|n} \left(1 - \frac{1}{p^r}\right) \geq \frac{1}{\zeta(r)} > 0.$$

Thus, there exists a constant $C > 0$ such that

$$|A'_n| \geq C a_n^r. \tag{1}$$

We have:

$$|A'_m \cap A'_n| = \sum_{\mathbf{k},\mathbf{l}} \left| \left(U\left(\frac{a_m}{m}\right) + \frac{\mathbf{k}}{m}\right) \cap \left(U\left(\frac{a_n}{n}\right) + \frac{\mathbf{l}}{n}\right) \right|,$$

where the summations are taken over all lattice vectors $\mathbf{k}, \mathbf{l}$ such that $\mathbf{k} \in U(m), (\mathbf{k}, m) = 1$; $\mathbf{l} \in U(n), (\mathbf{l}, n) = 1$. Apply a scale factor of $c = \frac{mn}{g}$, and write $g := (m, n)$. By the scaling property of Lebesgue measure, we then have:

$$\begin{aligned}
|A'_m \cap A'_n| &= \frac{1}{c^r} \sum_{\mathbf{k},\mathbf{l}} \left| \left(U\left(c \cdot \frac{a_m}{m}\right) + \frac{c\mathbf{k}}{m}\right) \bigcap \left(U\left(c \cdot \frac{a_n}{n}\right) + \frac{c\mathbf{l}}{n}\right) \right| \qquad (2) \\
&= \frac{1}{c^r} \sum_{\mathbf{k},\mathbf{l}} \left| \left(U\left(\frac{n}{g} a_m\right) + \frac{n\mathbf{k}}{g}\right) \bigcap \left(U\left(\frac{m}{g} a_n\right) + \frac{m\mathbf{l}}{g}\right) \right| \qquad (3) \\
&= \left(\frac{(m,n)}{mn}\right)^r \sum_{\mathbf{k},\mathbf{l}} \left| U\left(\frac{n}{(m,n)} a_m\right) \bigcap \left(U\left(\frac{m}{(m,n)} a_n\right) + \frac{m\mathbf{l} - n\mathbf{k}}{(m,n)}\right) \right|, \qquad (4)
\end{aligned}$$

where the last equality follows from Lemma 13 with $X = \frac{n\mathbf{k}}{g}$ and $Y = \frac{m\mathbf{l}}{g}$, and translation invariance of the measure. For every pair of $\mathbf{k}, \mathbf{l}$, we can write $m\mathbf{l} - n\mathbf{k} = (m,n)\mathbf{j}$, where $\mathbf{j}$ is a non-zero lattice vector (if $\mathbf{j} = \mathbf{0}$, then the coprimality conditions would force $m = n$, contradicting the standing assumption $m \neq n$). We wish to express the summation in (4) in terms of $\mathbf{j}$, so we have to count the number of pairs $\mathbf{k}, \mathbf{l}$ which produce the same $\mathbf{j}$. But note that $m\mathbf{l}' - n\mathbf{k}' = m\mathbf{l} - n\mathbf{k} \implies \mathbf{l}' \equiv \mathbf{l} \pmod{\frac{n}{(m,n)}}$. Thus, $\mathbf{j}$ takes on any given value at most $(m,n)^r$ times as $\mathbf{k}, \mathbf{l}$ ranges. Thus, we have the upper bound

$$\begin{aligned}
|A'_m \cap A'_n| &\leq \left(\frac{(m,n)^2}{mn}\right)^r S'\left(\frac{n}{(m,n)}a_m, \frac{m}{(m,n)}a_n\right) \\
&= \left(\frac{(m,n)^2}{mn}\right)^r O\left(\frac{n^r}{(m,n)^r}a_m^r \cdot \frac{m^r}{(m,n)^r}a_n^r\right) \\
&= \left(\frac{(m,n)^2}{mn}\right)^r O\left(\frac{n^r m^r a_m^r a_n^r}{(m,n)^{2r}}\right) \\
&= O(a_m^r a_n^r),
\end{aligned}$$

by applying Lemma 12 and absorbing constants into the Big-O. But this implies $|A'_m \cap A'_n| = O(|A'_m||A'_n|)$, which follows from (1). □

Before completing the proof of the higher-dimensional theorem, let us give an analogous result in the one-dimensional case *without monotonicity*, but restricted to *coprime approximations.*

Theorem 15. *Let $r = 1$. Suppose that $\sum_{n=1}^{N}|A'_n| \gg \sum_{n=1}^{N}|A_n|$, and that $\sum_{n=1}^{\infty}|A_n| = \infty$. Then, for almost every $x \in [0,1)$, there exist infinitely many pairs of integers p, q with $(p,q) = 1$ so that $|qx - p| < \alpha_q$.*

In other words, this result states that when we drop monotonicity, we require that the total measure of approximations using coprime fractions is **at least asymptotically a constant proportion** of the total measure of all approximations. Indeed, when we drop monotonicity, situations like the earlier described Duffin–Schaeffer counterexample can happen — we could have $\sum|A_n| = \infty$, but the sets A_n overlap so heavily that their union is tiny. In such pathological cases, almost no x is approximated.

Thus, the condition $\sum|A'_n| \gg \sum|A_n|$ ensures that this "overlap catastrophe" cannot happen, by forcing the "good approximations" (with reduced denominators) to contribute significantly to the total measure. The utility of Theorem 15 presents itself in a later section, where we will use it to prove an "almost all" version of the classical *Littlewood conjecture.*

Proof. We will proceed via the **second-moment method**. Let $E := \limsup_{n\to\infty} A'_n = \bigcap_{M=1}^{\infty}\bigcup_{n=M}^{\infty} A'_n$. For fixed M and $N \geq M$, define

$$\nu_{M,N}(x) := \sum_{n=M}^{N} 1_{A'_n}(x), \qquad B_{M,N} := \{x : \nu_{M,N}(x) > 0\} = \bigcup_{n=M}^{N} A'_n.$$

Then, $\mathbb{E}[\nu_{M,N}] = \displaystyle\int_0^1 \nu_{M,N}(x)\,\mathrm{d}x = \sum_{n=M}^{N}|A'_n|$. Also, $\mathbb{E}[\nu_{M,N}^2] = \sum_{m,n=M}^{N}|A'_m \cap A'_n|$. Now, performing a similar counting as the higher-dimensional case, with $g = \gcd(m,n)$, we can write:

$$|A'_m \cap A'_n| = \frac{g}{mn}\sum_{k,l}\left|U\left(\frac{n}{g}a_m\right)\bigcap\left(U\left(\frac{m}{g}a_n\right) + \frac{m\mathbf{l} - n\mathbf{k}}{g}\right)\right| \leq \frac{g^2}{mn}S'\left(\frac{n}{g}a_m, \frac{m}{g}a_n\right).$$

By Lemma 12, $S'\left(\frac{n}{g}a_m, \frac{m}{g}a_n\right) = O\left(\frac{n}{g}a_m \cdot \frac{m}{g}a_n\right)$, thus

$$|A'_m \cap A'_n| \leq \frac{g^2}{mn}O\left(\frac{n}{g}a_m \cdot \frac{m}{g}a_n\right) = O(a_m a_n).$$

As such, we have the overlap estimate $|A'_m \cap A'_n| \ll a_m a_n = |A_m||A_n|$ for $r = 1$. Using this overlap estimate, we obtain

$$\mathbb{E}[\nu_{M,N}^2] \ll \sum_{n=M}^{N}|A'_n| + \left(\sum_{n=M}^{N}|A_n|\right)^2.$$

Since $\sum_{n\leq N}|A'_n| \gg \sum_{n\leq N}|A_n|$ by assumption, it follows that for each fixed M and all sufficiently large N, $\sum_{n=M}^{N}|A_n| \ll \sum_{n=M}^{N}|A'_n|$. Hence, $\mathbb{E}[\nu_{M,N}^2] \ll \left(\sum_{n=M}^{N}|A'_n|\right)^2 = \left(\mathbb{E}[\nu_{M,N}]\right)^2$. By *Cauchy–Schwarz*,

$$\left(\mathbb{E}[\nu_{M,N}]\right)^2 = \left(\int_{B_{M,N}} \nu_{M,N}(x)\,\mathrm{d}x\right)^2 \leq |B_{M,N}|\,\mathbb{E}[\nu_{M,N}^2].$$

Thus, $|B_{M,N}| \geq \frac{\left(\mathbb{E}[\nu_{M,N}]\right)^2}{\mathbb{E}[\nu_{M,N}^2]} \geq c > 0$, for some absolute constant c, provided N is sufficiently large. Letting $N \to \infty$, we obtain $|\bigcup_{n=M}^{\infty} A'_n| \geq c$. Since the sets $\bigcup_{n=M}^{\infty} A'_n$ decrease in M, continuity from above gives $|E| = |\bigcap_{M=1}^{\infty}\bigcup_{n=M}^{\infty} A'_n| \geq c > 0$. Finally, **Gallagher's zero-one law** (see Theorem 1 of [Gal61]) implies that $\lambda(E) = 1$. $\square$

3.2. **Concluding the proof of Gallagher's theorem**

Let $T'_N(\mathbf{x})$ denote the number of solutions $n, \mathbf{l}$ to $n\mathbf{x} - \mathbf{l} \in U(a_n)$ with $(\mathbf{l}, n) = 1$. Define $S'_N = \sum_{n \le N} |A'_n|$. We have:

$$\begin{aligned}
\int_{U(1)} T'_N(\mathbf{x})^2 &= \int_{U(1)} \left(\sum_{n \le N} 1_{A'_n}(x) \right)^2 \quad \text{(For a fixed } n\text{, } l \text{ is unique)} \\
&= \int_{U(1)} \sum_{n,m \le N} 1_{A'_n}(x) 1_{A'_m}(x) \\
&= \sum_{n,m \le N} \int_{U(1)} 1_{A'_n}(x) 1_{A'_m}(x) \quad \text{(Fubini-Tonelli)} \\
&= \sum_{m,n \le N} |A'_m \cap A'_n| \\
&= \sum_{m=n} |A'_m \cap A'_n| + \sum_{m \ne n} |A'_m \cap A'_n| \\
&= \sum_{n} |A'_n| + \sum_{m \ne n} O(|A'_m||A'_n|) \\
&= S'_N + O(\sum_{m \ne n} |A'_m||A'_n|) \\
&= S'_N + O((S'_N)^2).
\end{aligned}$$

Thus, for $|S'_N| \ge 1$, we have that $\displaystyle\int_{U(1)} T'_N(\mathbf{x})^2 = O((S'_N)^2)$. We also have:

$$\begin{aligned}
\int_{U(1)} T'_N(\mathbf{x}) &= \int_{U(1)} \sum_{n \le N} 1_{A'_n}(\mathbf{x}) \\
&= \sum_{n \le N} \int_{U(1)} 1_{A'_n}(\mathbf{x}) \\
&= \sum_{n \le N} |A'_n| = S'_N.
\end{aligned}$$

We are now ready to prove the divergence case in Theorem 7. Let $E(K)$ be the set of all $\mathbf{x} \in U(1)$ for which $T'_N(\mathbf{x}) \ge K$, with a sufficiently large N.

Then,

$$S'_N = \int_{U(1)} T'_N(\mathbf{x}) = \int_{U(1) \setminus E(K)} T'_N(\mathbf{x}) + \int_{E(K)} T'_N(\mathbf{x}).$$

For the first integral, note that $T'_N(\mathbf{x}) < K$, so that $\int_{U(1)\setminus E(K)} T'_N(\mathbf{x}) < K|U(1) \setminus E(K)| = K(1 - |E(K)|)$. For the second integral, by the *Cauchy–Schwarz inequality*, we have:

$$\int_{E(K)} T'_N(\mathbf{x}) \leq \left(\int_{E(K)} T'_N(\mathbf{x})^2\right)^{1/2} \left(\int_{E(K)} 1\right)^{1/2} = \left(|E(K)| \int_{E(K)} T'_N(\mathbf{x})^2\right)^{1/2}.$$

Thus, combining these estimates, we have:

$$S'_N \leq (1 - |E(K)|)K + \left(|E(K)| \int_{E(K)} T'_N(\mathbf{x})^2\right)^{1/2}.$$

Now, note that $S'_N > C \cdot S_N$ for some $C > 0$, where $S_N \to \infty$ as $N \to \infty$ by the assumption of divergence. Thus, $S'_N \to \infty$. Dividing the preceding inequality by S'_N, we obtain:

$$1 \leq \frac{(1 - |E(K)|)K}{S'_N} + \frac{|E(K)|^{1/2}\, O(S'_N)}{S'_N},$$

where we have also replaced the integral of $T'_N(\mathbf{x})^2$ with the corresponding asymptotic expression. Fixing K and letting $N \to \infty$, we thus have $|E(K)| \geq C_1$, where the positive constant C_1 appears as a result of the asymptotic expression. Thus, letting $K \to \infty$, we see that the set $\{\mathbf{x} : n\mathbf{x} - \mathbf{l} \in U(a_n),\ (\mathbf{l}, n) = 1 \text{ has infinitely many solutions}\}$ has positive measure. Our goal is then to upgrade this positive measure to full measure. This follows from well-known results in Ergodic theory (see Lemma 19 below).

Definition 16 (Ergodicity)**.** A transformation σ is ergodic *if and only if* every invariant set has measure 0 or 1. Here, a set S is said to be invariant if $S = \sigma S$ a.e.

Proposition 17. *A measure-preserving transformation σ is ergodic if and only if every invariant measurable function (that is, $f(\sigma x) = f(x)$ a.e.) is constant a.e.*

Proposition 18 (Toral Automorphisms)**.** *An automorphism of the torus is ergodic if and only if no eigenvalue is a root of unity.*

Identify $U(1)$ with the torus $\mathbb{T}^r = \mathbb{R}^r/\mathbb{Z}^r$ by taking fractional parts. For $c \leq 1$, we thus regard $U(c)$ as a subset of $\mathbb{T}^r$. Replace a_n with b_n, where $b_n = o(a_n)$, and $\sum b_n^r$ diverges. Let E^* be the set of all $\mathbf{x} \in \mathbb{T}^r$ for which

$$\mathbf{x} - \mathbf{x}_n \in U\left(\frac{b_n}{n}\right),\ \mathbf{x}_n \text{ of order } n \text{ in } \mathbb{T}^r$$

has infinitely many solutions $n, \mathbf{x_n}$. Thus, $\mathbf{x}_n$ simply consists of all rational points such that $nx_n \equiv 0 \pmod 1$. By the preceding paragraph, we see that E^* has positive measure.

Lemma 19 (Existence of Ergodic Automorphisms)**.** *Let $r \geq 2$. Then, there exists an ergodic automorphism σ of the torus $\mathbb{T}^r$ for which $\sigma U(c) \subseteq U(rc)$ for $rc \leq 1$.*

Proof. See Lemma 2 of [Gal65]. $\square$

We now have all the ergodicity machinery needed to complete the proof of Theorem 7. Let E denote the set of $\mathbf{x} \in \mathbb{T}^r$ for which

$$\mathbf{x} - \mathbf{x}_n \in U\left(\frac{a_n}{n}\right), \quad \operatorname{ord}(\mathbf{x}_n) = n$$

for infinitely many pairs $(n, \mathbf{x}_n)$. We claim that

$$\sigma^k(E^*) \subseteq E \quad \text{for every fixed } k \geq 0.$$

Indeed, let $\mathbf{x} \in E^*$. For infinitely many n, there is a point $\mathbf{x}_n$ of exact order n such that

$$\mathbf{x} - \mathbf{x}_n \in U\left(\frac{b_n}{n}\right).$$

Since a group automorphism preserves the order of a point, $\sigma^k \mathbf{x}_n$ also has exact order n. Moreover, for all sufficiently large such n, repeated application of Lemma 19 gives

$$\sigma^k \mathbf{x} - \sigma^k \mathbf{x}_n \in \sigma^k U\left(\frac{b_n}{n}\right) \subseteq U\left(\frac{r^k b_n}{n}\right).$$

For fixed k, the relation $b_n = o(a_n)$ implies that $r^k b_n \leq a_n$ for all sufficiently large n. Hence,

$$\sigma^k \mathbf{x} - \sigma^k \mathbf{x}_n \in U\left(\frac{a_n}{n}\right)$$

for infinitely many n, proving that $\sigma^k \mathbf{x} \in E$. Now, set

$$F := \bigcup_{k=0}^{\infty} \sigma^k(E^*).$$

Then, $F \subseteq E$ and $\lambda(F) > 0$. Furthermore, $\sigma(F) \subseteq F$. Since σ is measure-preserving, $\lambda(\sigma(F)) = \lambda(F)$, and therefore $\sigma(F) = F$ up to a null set. By ergodicity, $\lambda(F) \in \{0, 1\}$. Since $\lambda(F) > 0$, we have $\lambda(F) = 1$, and consequently $\lambda(E) = 1$. This completes the divergence case of Theorem 7.

4. General Duffin–Schaeffer-type counterexamples

We now return to the one-dimensional case. In a recent paper, Chow et al. [CHPR25] proved the following result:

Theorem 20. *Let $f : \mathbb{N} \to [0, \infty)$ be a non-increasing function such that* $\lim_{q\to\infty} f(q) =$

0 and $\sum_{q\geq 1} f(q) = \infty$*. Then, there exists* $S \subseteq \mathbb{N}$ *such that*

$$\sum_{q\in S} f(q) = \infty, \quad \lambda(A(f\mathbf{1}_S)) = 0.$$

Here, λ is the Lebesgue measure and the set A is as defined in Khintchine's theorem (Theorem 1). In fact, the authors also proved the following refinement:

Theorem 21. *Let* $f : \mathbb{N} \to [0,\infty)$ *be a non-increasing function such that* $\lim_{q\to\infty} f(q) = 0$ *and* $\sum_{q\geq 1} f(q) = \infty$*. Then there exists* $\psi : \mathbb{N} \to [0,\infty)$*, non-increasing on its support, such that*

$$\psi \leq f, \quad \sum_{q\in\mathbb{N}} \psi(q) = \infty,\ \lambda(A(\psi)) = 0,\ \overline{\delta}(\text{supp } \psi) = 1.$$

Here, $\overline{\delta}(S) := \limsup_{N\to\infty} \dfrac{|S\cap[1,N]|}{N}$ is the *upper density* of $S \subseteq \mathbb{N}$. Thus, this result says that simply asking for functions which are monotonic on a support that has full density infinitely often is insufficient for guaranteeing Khintchine's theorem.

Now, to prove Theorem 21, a natural first attempt is to take $\psi = 1_S f$, for some $S \subseteq \mathbb{N}$ of full upper density. However, this approach fails. Indeed, let $f(q)$ be sufficiently large, in the precise sense of $f(q) \gg \frac{1}{\varphi(q)}$, where φ is the Euler totient function. (The motivation behind this size constraint is for us to appeal to the full power of the Duffin–Schaeffer conjecture). Suppose $\overline{\delta}(S) = 1$, and let $\epsilon \in (0, \frac{1}{2})$. Let $Q \geq 1$ such that

$$\frac{\#([1,Q]\cap\mathbb{N})\setminus S}{Q} < \epsilon,$$

i.e. we want the number of missing integers from S to be smaller than ϵQ. Then,

$$\begin{aligned}\sum_{q\in S}\frac{\varphi(q)f(q)}{q} &\geq \sum_{q\in S\cap[1,Q]}\frac{\varphi(q)f(q)}{q} \geq \sum_{q\in S\cap[1,Q]}\frac{\varphi(q)}{q}\frac{1}{\varphi(q)} \\ &= \sum_{q\in S\cap[1,Q]}\frac{1}{q} \\ &\geq \sum_{\epsilon Q<q\leq Q}\frac{1}{q},\end{aligned}$$

where the last inequality follows since we removed at most $\lfloor \epsilon Q\rfloor$ terms of small denominators q. Moreover, $\sum_{\epsilon Q<q\leq Q}\frac{1}{q} \gg \ln(\epsilon^{-1})$, since $\displaystyle\int_{\epsilon Q}^{Q}\frac{1}{q} = [\ln q]_{\epsilon Q}^{Q} = \ln\left(\frac{Q}{\epsilon Q}\right) = \ln(\epsilon^{-1})$. Taking $\epsilon \to 0$, we see that $\sum_{q\in S}\frac{\varphi(q)f(q)}{q} = \infty$. But by the *Duffin–Schaeffer conjecture* (in the sense of Koukoulopoulos–Maynard; see [KM20]), we have that $\lambda(A(f\mathbf{1}_S)) = 1$.

To prove Theorem 20, we first need the following proposition ([CHPR25]) as a black box, which constructs a finite block of denominators S satisfying some conditions. Here, we only give a heuristic explanation of the construction after the statement.

Proposition 22. *Let $f : \mathbb{N} \to [0, \infty)$ be a non-increasing function satisfying $\lim_{q\to\infty} f(q) = 0$, $\sum_{q=1}^{\infty} f(q) = \infty$. Set*

$$A_q = \{\alpha \in [0,1] : \|q\alpha\| < f(q)\}.$$

Then, for any $\epsilon, M > 0$, there exists a finite set $S \subseteq \mathbb{N} \cap (M, \infty)$ such that

$$\sum_{q\in S} \lambda(A_q) \in [1,2], \quad \lambda\left(\bigcup_{q\in S} A_q\right) < \epsilon.$$

Essentially, although the total expected measure of the approximation sets is around 1, the measure of their union can be made arbitrarily small, forming the basic building blocks of the counterexample.

1. (**Constructing multiplicatively structured denominators**:) The denominators are chosen in product form $S = X \cdot Y$. Here, Y is highly multiplicative and consists of products of specific prime pairs, so that elements of Y frequently share large common divisors. X consists of much larger integers from an arithmetic progression, chosen to control the divergence of the series.

2. (**Controlling overlaps via number-theoretic structure**:) The main enemy in the argument lies in the possibility of the sets $A_q := \{\alpha \in [0,1] : \|q\alpha\| < f(q)\}$ overlapping heavily when denominators share common factors. To show that $\lambda(\bigcup_{q\in S} A_q)$ is small, the proof relies on bounding the measure of overlaps between sets A_{xy} and $A_{xy'}$ for $y, y' \in Y$. The authors establish that the measure of the union over Y is bounded by a weighted sum $\lambda(\bigcup_{y\in Y} A_{xy}) \leq 2\sum_{y\in Y} g_{\mathcal{I}}(y)f(xy)$. Here, the set $\mathcal{I}$ refers to pairs of primes (p_{2i-1}, p_{2i}), such that for a given integer j, the primes fall within $[e^j, e^{j+1})$, and $p_{2i} - p_{2i-1} \leq 40j$. $\mathcal{I}$ is then the union of these index sets over the different scales j, from 5 to some large integer K. $g_{\mathcal{I}}(y)$ thus acts as a "savings factor", representing the proportion of the set A_{xy} *not* already covered by earlier sets in the union.

3. (**Probabilistic Methods**:) It is necessary to prove that $g_{\mathcal{I}}(y)$ is small for most elements in Y. The proof treats the indices of primes dividing $y \in Y$ as random variables $X_j(y)$ under a uniform probabilistic measure on Y. By appealing to *Hoeffding's Inequality*, it is shown that these variables concentrate around their

mean, so that only a negligible subset of Y, denoted by $Y(\mathcal{I},\delta)$, has large values of $g_{\mathcal{I}}(y)$. This ensures that the weighted measure of the union remains below the target threshold ϵ.

4. (**Inheritance of Monotonicity:**) Finally, the proof uses the fact that the approximation function f is non-increasing to compare the discrete sum over X with an integral. The authors define an auxiliary function $F(y)$ that inherits the monotonicity of f. They then tune the size of $x_{\max}$ so that $\sum_{q\in S}\lambda(A_q)$ falls within $[1,2]$ while the probabilistic bounds on $g_{\mathcal{I}}(y)$ keep the measure of the union small.

Proof of Theorem 20. The idea is to construct a sequence of pairwise disjoint sets $\{S_n\}$ recursively using Proposition 22. Choose $M = 1$ and $\epsilon = \frac{1}{2}$, and obtain an S_1 satisfying the hypotheses of Proposition 22 (here, S_1 plays the role of S). Now, suppose we have constructed $S_1, \cdots, S_{k-1}$. Set $\epsilon = 2^{-k}$ and $M_k := \max S_{k-1}$, the maximum element of S_{k-1}. Thus, by Proposition 22, S_k as defined is disjoint from all of $S_1, S_2, \cdots, S_{k-1}$. We define $S := \bigcup_{k\in\mathbb{N}} S_k$, our desired candidate for S in Theorem 20. Clearly,

$$\sum_{q\in S}\lambda(A_q) = \sum_{k\in\mathbb{N}}\sum_{q\in S_k}\lambda(A_q) = \infty,$$

since the Lebesgue measure of each $\sum_{q\in S_k}\lambda(A_q)$ is at least 1. Let a be the closest integer to αq. Then, $\alpha \in A_q \implies -\frac{f(q)}{q} + \frac{a}{q} < \alpha < \frac{f(q)}{q} + \frac{a}{q}$, with interval length $\frac{2f(q)}{q}$. As a ranges over $0, 1\cdots, q$, we see that

$$\lambda(A_q) \le (q+1)\frac{2f(q)}{q} = 2f(q) + \frac{2f(q)}{q} \le 4f(q).$$

Thus, $\sum_{q\in S}\lambda(A_q) = \infty \implies \sum_{q\in S} f(q) = \infty$. On the other hand,

$$\begin{aligned} A(f\mathbf{1}_S) \subseteq \bigcup_{l\ge k}\bigcup_{q\in S_l} A_q &\implies \lambda(A(f\mathbf{1}_S)) \le \sum_{l\ge k}\lambda\left(\bigcup_{q\in S_l} A_q\right) \\ &\implies \lambda(A(f\mathbf{1}_S)) \le \lim_{k\to\infty}\sum_{l\ge k}\lambda\left(\bigcup_{q\in S_l} A_q\right) \le \lim_{k\to\infty}\sum_{l\ge k} 2^{-l} = 0. \end{aligned}$$

□

Proof of Theorem 21. We will choose ψ to be supported on pairwise disjoint blocks of the form $C_j \cup D_j$, so that $\operatorname{supp}(\psi) = \bigcup_{j=1}^{\infty}(C_j \cup D_j)$. Here, C_j are sparse blocks with significant contributions to $\sum\psi(q)$, and the D_j are dense with negligible contributions to $\sum\psi(q)$. The construction proceeds as follows:

1. Let the dense block D_{j-1} end at Q_{j-1}.

2. Choose a gap $M_j > Q_{j-1}$.

3. Choose a sparse block C_j from integers larger than M_j.

4. Let D_j start after C_j ends and go up to a new large integer Q_j.

$$\cdots < Q_{j-1} < \cdots < C_j < D_j < \cdots < C_{j+1} \cdots$$

First, assign $D_0 = \{1\}$, $Q_0 = 1$, and $\psi(1) = f(1)$. (clearly $f(1) \neq 0$, else $\sum f(i)$ would not diverge). For each $j \geq 1$, we have $\psi(Q_{j-1}) > 0$, and choose $M_j > Q_{j-1}$ large enough such that $f(M_j) \leq \psi(Q_{j-1})$ (again $f(M_j) \neq 0$ for the same reason). By Proposition 22, noting that $\lambda(A_q) = 2f(q)$ for sufficiently large q, we may find a finite set $C_j \subset \mathbb{N} \cap (M_j, \infty)$ such that

$$\sum_{q \in C_j} f(q) \in [\frac{1}{2}, 1], \quad \lambda\left(\bigcup_{q \in C_j} A_q\right) < 2^{-j}.$$

Now, let $D_j := (\max C_j, Q_j] \cap \mathbb{N}$, where Q_j is sufficiently large such that $\frac{Q_j - \max C_j}{Q_j} \geq 1 - \frac{1}{j}$. Define ψ piecewise as follows:

$$\psi(q) = \begin{cases} f(q) \quad \text{if } q \in C_j \text{ for some } j \geq 1 \\ \min\{f(q), q^{-2}\} \quad \text{if } q \in D_j \text{ for some } j \geq 1 \\ 0 \quad \text{otherwise.} \end{cases}$$

Clearly, $\psi \leq f$ by construction, and $\text{supp}(\psi) = \bigcup_{j=1}^{\infty}(C_j \cup D_j)$ by construction. Moreover, ψ is monotonic on its support by construction. To compute $\overline{\delta}(\text{supp}(\psi))$, we look at each of the quantities $\frac{|\text{supp}(\psi) \cap [1,N]|}{N}$. At $N = Q_j$, we have contained the entire set D_j. Thus, the count of elements in the support up to Q_j is at least $|D_j| = Q_j - \max C_j$. Hence,

$$\begin{aligned} \frac{|\text{supp}(\psi) \cap [1, Q_j]|}{Q_j} &\geq \frac{Q_j - \max C_j}{Q_j} \\ &\geq 1 - \frac{1}{j} \\ &\to 1 \quad (\text{taking } j \to \infty). \end{aligned}$$

This proves that $\mathrm{supp}(\psi)$ has upper density 1. Thus,

$$\sum_{q=1}^{\infty} \psi(q) \geq \sum_{j=1}^{\infty} \sum_{q \in C_j} \psi(q) = \sum_{j=1}^{\infty} \sum_{q \in C_j} f(q) = \infty,$$

where the last equality follows since we assumed that each $\sum_{q \in C_j} f(q) \geq \frac{1}{2}$. It remains to show that $\lambda(A(\psi)) = 0$, which follows from the **First Borel–Cantelli lemma**. Specifically, $\sum \lambda\left(\bigcup_{q \in C_j} A_q\right) < \sum 2^{-j} < \infty$, so almost no x lies in infinitely many of the sets $\bigcup_{q \in C_j} A_q$. Furthermore,

$$\sum_{j=1}^{\infty} \sum_{q \in D_j} \psi(q) \leq \sum_{j=1}^{\infty} \sum_{q \in D_j} \frac{1}{q^2} \leq \sum \frac{1}{q^2} < \infty,$$

so that $\displaystyle\sum_{q \in \bigcup D_j} \lambda(A_q) = \sum_{j=1}^{\infty} \sum_{q \in D_j} \lambda(A_q) \approx \sum_{j=1}^{\infty} \sum_{q \in D_j} 2\psi(q) < \infty$, i.e. almost no x lies in infinitely many of the sets A_q with $q \in D_j$. Thus, $\lambda(A(\psi)) = 0$, as desired. $\square$

We note that the choice of q^{-2} in the piecewise definition of ψ is nothing special — one could have replaced it with another non-increasing function $g(q)$ of sufficiently quick decay such that $\sum g(q) < \infty$, so that the First Borel–Cantelli lemma still works.

5. **The Littlewood conjecture and related results**

J.E. Littlewood proposed in 1930 the following conjecture — let $\alpha, \beta \in \mathbb{R}$. Then, is it true that $\liminf_{n\to\infty} n\|n\alpha\| \cdot \|n\beta\| = 0$? (Here, $\|x\|$ denotes the distance of x to the nearest integer.) In [Gal62], Gallagher proved the following stronger result for *almost every* $(\alpha, \beta) \in \mathbb{R}^2$.

Proposition 23. *For almost every $(\alpha, \beta) \in \mathbb{R}^2$ one has that*

$$\liminf_{n\to\infty} n(\log n)^2 \|n\alpha\| \cdot \|n\beta\| = 0.$$

In particular, of interest is *fibration* — what happens if we fix some $\alpha \in \mathbb{R}$ and allow β to vary? The following (inhomogeneous) result was established by *Beresnevich–Haynes–Velani* (see Section 2.2 of [BHV16]) —

Proposition 24. *Fix $\alpha \in \mathbb{R}$ such that α is irrational and not Liouville. Let $\gamma \in \mathbb{R}$. Assume further that the Duffin–Schaeffer conjecture holds. Then, for almost every $\beta \in \mathbb{R}$,*

$$\liminf_{n\to\infty} n(\log n)^2 \|n\alpha - \gamma\| \cdot \|n\beta\| = 0.$$

In the special case where $\gamma = 0$ (homogeneous case), we do not need to impose any additional assumptions on α, since we can establish the stronger result $\liminf_{n\to\infty} n^2\|n\alpha\| = 0$ in case α is rational or Liouville (recall that α is Liouville if $\liminf_{n\to\infty} n^\omega\|n\alpha\| = 0$ for all $\omega > 0$). Indeed, if α is rational, then $\|n\alpha\| = 0$ infinitely often implies that $n^2\|n\alpha\| = 0$ infinitely often, implying that $\liminf_{n\to\infty} n^2\|n\alpha\| = 0$. As for the case where α is Liouville, we can simply set $\omega = 2$.

We note that at the time this result was proved, the Duffin–Schaeffer conjecture had not yet been proved, so the result was conditional. Improving upon this result, what Chow did in [Cho18] was to prove the above result without invoking the full Duffin–Schaeffer conjecture. His argument instead used what he called the *"Duffin–Schaeffer theorem"* (in the sense of Theorem 15). In fact, Chow proved a stronger version of Proposition 24.

Proposition 25. *Let $\gamma \in \mathbb{R}$ and $\alpha \in \mathbb{R}$ be irrational and not Liouville. Let $\psi : \mathbb{N} \to \mathbb{R}_0^+$ be a monotone decreasing function, such that $\sum_{n\in\mathbb{N}} \psi(n)\log n = \infty$. Then, for almost every $\beta \in \mathbb{R}$, there exist infinitely many $n \in \mathbb{N}$ such that*

$$\|n\alpha - \gamma\| \cdot \|n\beta\| < \psi(n).$$

To prove Proposition 25, we define an auxiliary function $\Phi(n) = \dfrac{\psi(n)}{\|n\alpha - \gamma\|}$. It suffices to show that for almost every $\beta \in \mathbb{R}$, there exist infinitely many $n \in \mathbb{N}$ such that $\|n\beta\| < \Phi(n)$. We restrict the support of Φ to the complement of a "poorly-behaved" set B (to be properly defined later), and study the two sums $\sum_{n\le N, n\notin B} \dfrac{1}{\|n\alpha - \gamma\|}$ and $\sum_{n\le N, n\notin B} \dfrac{\psi(n)\varphi(n)}{n\|n\alpha - \gamma\|}$. For the first sum, we are interested in upper bounds, and for the second sum, we are interested in lower bounds. To this end, the major results of Chow's paper are Lemmas 28 and 29 (see below).

Definition 26 (Bohr Sets)**.** Let $\alpha, \gamma \in \mathbb{R}$ such that α is irrational and not Liouville. For $\rho > 0$ and $N \in \mathbb{N}$, define a Bohr set

$$N_\gamma(\alpha, \rho) = \{n \in \mathbb{N} : n \le N, \|n\alpha - \gamma\| < \rho\}.$$

Definition 27 (Generalised Arithmetic Progression)**.** Define

$$P(b, A_1, A_2, N_1, N_2) := \{b + A_1 n_1 + A_2 n_2, 1 \le n_1 \le N_1, 1 \le n_2 \le N_2\}.$$

Recall that the *irrationality exponent* is defined by

$$\mu(\alpha) := \sup\{w > 0 : \|n\alpha\| < n^{-w} \quad \text{for infinitely many } n \in \mathbb{N}\}.$$

Set $\lambda := \mu(\alpha) + 1$. Since α is irrational, *Dirichlet's theorem* guarantees that $\lambda \ge 2$. Since α is not Liouville, $\lambda < \infty$. **For the rest of this section, set** $\epsilon := \dfrac{1}{10\lambda}$.

We are chiefly concerned with two issues:

1. Find a *Generalised Arithmetic Progression* (GAP) contained in an inhomogeneous Bohr set $N_\gamma(\alpha, \rho)$.
2. Show that $\dfrac{\psi(n)}{n} \gg 1$ on average over a GAP.

For (1), we have the following lemma:

Lemma 28 (Existence of GAPs in Bohr Sets)**.** *Let $N^{-2\epsilon} \leq \rho \leq N^{-\epsilon}$. Then, there exists $b, A_1, A_2, N_1, N_2 \in \mathbb{N}$ such that*

1. $P(b, A_1, A_2, N_1, N_2) \subseteq N_\gamma(\alpha, \rho)$
2. $N_1 N_2 \asymp \rho N$
3. $A_2 > N_1$
4. $\min(N_1, N_2) > N^\epsilon$
5. $\frac{N}{20} \leq b \leq \frac{N}{2}$.

Let $\varphi(n)$ denote Euler's totient function. For (2), we have the following lemma:

Lemma 29 (Averages of Totient)**.** *Define the restricted Bohr set, $N^*_\gamma(\alpha, \rho) := N_\gamma(\alpha, \rho) \cap [\frac{N}{20}, N]$. Let $N^{-2\epsilon} \leq \rho \leq N^{-\epsilon}$. Then,*

$$\sum_{n \in N^*_\gamma(\alpha,\rho)} \frac{\varphi(n)}{n} \asymp \rho N.$$

Before proving Lemma 28, we need several preliminary results. We first record some results regarding continued fractions.

Proposition 30. *Let $\alpha \in \mathbb{R}$ be irrational. Then, α admits a simple continued fraction expansion; namely, one has:*

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}} \quad \textit{uniquely.}$$

Write $\alpha = [a_0 : a_1, a_2, \cdots,]$, the sequence of partial quotients. Finite truncations of α are called convergents — we write $\dfrac{p_k}{q_k} = [a_0 : a_1, a_2, \cdots, a_k]$ with $\gcd(p_k, q_k) = 1$. Then,

1. $q_0 = 1, q_1 = a_1,$ *and* $q_{k+1} = a_{k+1} q_k + q_{k-1}, k \geq 1$.

2. $|\alpha - \frac{p_k}{q_k}| < \frac{1}{q_k q_{k+1}}$.
3. $\|q_k\alpha\| < \frac{1}{q_{k+1}}$.
4. *For all* k, $\gcd(q_k, q_{k+1}) = 1$.
5. *The sequence of denominators* (q_k) *is strictly increasing.*

Proof. These are standard facts about continued fractions. □

Proposition 31 (Bounded Growth of Denominators)**.** *Let* $\alpha \in \mathbb{R}$ *be irrational and not Liouville. Then, there exists* $C > 0$ *such that* $q_{l+1} \leq C \cdot q_l^{\lambda}$ *for all* $l \in \mathbb{Z}_0^+$.

Proof. By definition of the irrationality exponent μ, for all but finitely many n, one has $\|n\alpha\| \geq c \cdot n^{-\mu}$ for some $c > 0$. Now, by item (3) of Proposition 30, one has $\|q_l\alpha\| < \dfrac{1}{q_{l+1}}$. Picking l sufficiently large so that q_l is also sufficiently large (i.e. $l \geq l_0$), this implies $\dfrac{1}{q_{l+1}} > \|q_l\alpha\| \geq c \cdot q_l^{-\mu}$. Thus,

$$q_{l+1} < c^{-1} q_l^{\mu} < c \cdot q_l^{\mu+1} = c \cdot q_l^{\lambda}.$$

Finally, setting $C = \max\left\{c^{-1}, \max_{l \leq l_0} \frac{q_{l+1}}{q_l^{\lambda}}\right\}$ completes the proof. □

Theorem 32 (Three Distance Theorem)**.** *Let* $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ *and* $m \in \mathbb{N}$. *Then,*

1. m *can be uniquely represented as* $m = rq_k + q_{k-1} + s$, *for some* $k \geq 0$, $1 \leq r \leq a_{k+1}$, *and* $0 \leq s < q_k$. *(Recall that the* a_k*'s are the partial quotients of* α*.)*
2. *Define* $\eta_k := |q_k\alpha - p_k|$. *For the partition* $P_\alpha(m)$ *defined by the fractional parts* $\{\alpha\}, \{2\alpha\}, \cdots, \{m\alpha\}$, *there are* $(r-1)q_k + q_{k-1} + s + 1$ *subintervals of length* η_k, $s+1$ *subintervals of length* $\eta_{k-1} - r\eta_k$, *and* $q_k - (s+1)$ *subintervals of length* $\eta_{k-1} - (r-1)\eta_k$, *where the unique integers* k, r *and* s *are as per (a).*
3. *The maximum length,* $d_\alpha(m)$, *and minimum length,* $d'_\alpha(m)$, *of the subintervals of partition* $P_\alpha(m)$ *are:*
   - *When* $q_k > s + 1$,

$$d_\alpha(m) = \begin{cases} \eta_{k+1} + \eta_k, & r = a_{k+1} \\ \eta_{k+1} + (a_{k+1} - r + 1)\eta_k, & r < a_{k+1} \end{cases}$$

   - *When* $q_k = s + 1$,

$$d_\alpha(m) = \begin{cases} \eta_k, & r = a_{k+1} \\ \eta_{k+1} + (a_{k+1} - r)\eta_k, & r < a_{k+1} \end{cases}$$

- *For all* $q_k \geq s+1$,

$$d'_\alpha(m) = \begin{cases} \eta_{k+1}, & r = a_{k+1} \\ \eta_k, & r < a_{k+1} \end{cases}$$

Proof. See [MK98, Theorem 1 and Corollary 1]. □

The proof of Lemma 28 has three steps, which we outline as follows:

1. Find a base point b_0 where $\|b_0\alpha - \gamma\| \leq \frac{\rho}{10}$(handles the inhomogeneous shift γ).
2. Construct a GAP in a homogeneous Bohr set.
3. Shift the base point and verify inclusion.

Proof of Lemma 28. We first claim that there exists $b_0 \in \mathbb{N}$ with $b_0 \leq \frac{N}{10}$ such that $\|b_0\alpha - \gamma\| \leq \frac{\rho}{10}$. Indeed, set $M := \lfloor \frac{N}{10} \rfloor$. Define $d_1, \cdots, d_M := \{\{n\alpha\} : 1 \leq n \leq M\}$, with $0 = d_0 < d_1 < \cdots < d_M < d_{M+1} = 1$. Now, $\{\gamma\} \in [d_i, d_{i+1}]$ for some i. Also, there exists $b_0 \in \{1, 2, \cdots, M\}$ such that $\{b_0\alpha\}$ is either d_i or d_{i+1}. Then,

$$\|b_0\alpha - \gamma\| \leq d_{i+1} - d_i \leq \max_{0 \leq i \leq M} (d_{i+1} - d_i) := d_\alpha(M).$$

By item (1) of the *Three Distance theorem* (Theorem 32), we have, uniquely, $M = rq_k + q_{k-1} + s$, $1 \leq r \leq a_{k+1}, 0 \leq s < q_k$. In addition, by item (3) of the same Theorem, we have the upper bound $d_\alpha(M) \ll a_{k+1}\|q_k\alpha\| \ll \frac{a_{k+1}}{q_{k+1}} \ll \frac{1}{q_k}$.

Thus, by Proposition 31,

$$\begin{aligned} \|b_0\alpha - \gamma\| &\leq d_\alpha(M) \\ &\ll \frac{1}{q_k} \\ &\ll q_{k+1}^{-\frac{1}{\lambda}} \quad \text{(By Proposition 31)} \\ &\ll M^{\frac{-1}{\lambda}} \quad \text{(Since } q_k \leq M) \\ &\ll N^{-3\epsilon} \\ &\ll N^{-2\epsilon}. \end{aligned}$$

Thus, $\|b_0\alpha - \gamma\| \leq \frac{\rho}{10}$, as claimed.

To continue with the second step as outlined, choose $t \in \mathbb{N}$ such that

$$\rho q_{t-1}^2 \leq N < \rho q_t^2 \implies q_t > \sqrt{\frac{N}{\rho}} > \sqrt{N}.$$

Depending on our choice of N, define the pair (A_1, A_2) as follows:

$$(A_1, A_2) := \begin{cases} (q_{t-2}, q_{t-1}), & \text{if } N < \rho q_{t-1} q_t \\ (q_{t-1}, q_t), & \text{if } N \geq \rho q_{t-1} q_t. \end{cases}$$

In both cases, Proposition 30 implies that $\gcd(A_1, A_2) = 1$. Define the candidates $N_1 := \lfloor \frac{\rho A_2}{10} \rfloor, N_2 := \lfloor \frac{N}{10 A_2} \rfloor$. By construction, $A_2 > N_1$, and we argue explicitly in both cases.

In the first case where $N < \rho q_{t-1} q_t$,

$$\rho A_2 = \rho q_{t-1} \gg \rho q_t^{\frac{1}{\lambda}} \gg \rho N^{\frac{1}{2\lambda}} \gg N^{2\epsilon}.$$

Also, since $N \geq \rho q_{t-1}^2$, we have that $\frac{N}{A_2} = \frac{N}{q_{t-1}} \geq \rho q_{t-1} \gg N^{2\epsilon}$. Thus, $\min(N_1, N_2) \geq N^{\epsilon}$, and $N_1 N_2 \asymp \frac{\rho A_2}{10} \times \frac{N}{10 A_2} \asymp \rho N$. The other case with $N \geq \rho q_{t-1} q_t$ is analogous.

Finally, to proceed with the third step of the proof as outlined, define $b := b_0 + N_2 A_2$. Then, $b \leq \frac{N}{10} + \frac{N}{10} = \frac{N}{5} < \frac{N}{2}$ and $b \geq N_2 A_2 = \lfloor \frac{N}{10 A_2} \rfloor A_2 > \frac{1}{2} \frac{N}{10 A_2} A_2 = \frac{N}{20}$, so b lies in the claimed range. It remains to prove item (1) of the Lemma. For any $n = b + A_1 n_1 + A_2 n_2 \in P(b, A_1, A_2, N_1, N_2)$ with $1 \leq n_1 \leq N_1, 1 \leq n_2 \leq N_2$, we have:

$$n \leq b + A_1 n_1 + A_2 N_2 \leq \frac{N}{5} + \frac{\rho A_1 A_2}{10} + \frac{N}{10} < N.$$

Now, by the triangle inequality, and substituting b in terms of b_0, we obtain:

$$\begin{aligned} \|n\alpha - \gamma\| &= \|(b_0 + N_2 A_2 + A_1 n_1 + A_2 n_2)\alpha - \gamma\| \\ &\leq \|b_0 \alpha - \gamma\| + N_1 \|A_1 \alpha\| + 2N_2 \|A_2 \alpha\| \\ &\leq \frac{\rho}{10} + N_1 \|A_1 \alpha\| + 2N_2 \|A_2 \alpha\|. \end{aligned}$$

If $N < \rho q_{t-1} q_t$, then $\|A_1 \alpha\| = \|q_{t-2} \alpha\| \leq \frac{1}{q_{t-1}}$, by item (3) of Proposition 30. Similarly, $\|A_2 \alpha\| = \|q_{t-1} \alpha\| \leq \frac{1}{q_t}$. Thus,

$$\begin{aligned} N_1 \|A_1 \alpha\| &\ll N_1 q_{t-1}^{-1} \\ &\ll \frac{\rho A_2}{10} q_{t-1}^{-1} \\ &= \frac{\rho q_{t-1} q_{t-1}^{-1}}{10} \\ &= \frac{\rho}{10}. \end{aligned}$$

Similarly,

$$\begin{aligned} 2N_2 \|A_2 \alpha\| &\ll 2 \cdot \frac{N}{10 A_2} q_t^{-1} \\ &= \frac{N}{5 q_{t-1} q_t} \end{aligned}$$

$$\ll \frac{N}{5q_t^2}$$
$$\leq \frac{\rho}{5} \quad (\text{recall } N < \rho q_t^2).$$

Combining these estimates gives

$$\|n\alpha - \gamma\| \leq \frac{\rho}{10} + \frac{\rho}{10} + \frac{\rho}{5} \leq \rho,$$

proving that the GAP lies in the Bohr set. The other case where $N \geq \rho q_{t-1} q_t$ is analogous. $\square$

Before we prove Lemma 29, we need the following counting result.

Lemma 33. *Let $S \subseteq \mathbb{R}^2$ be a convex set containing the origin, and suppose S lies in the compact disc of radius r centred at the origin. Let $V(S)$ denote the area of S, and let $\Lambda \subseteq \mathbb{Z}^2$ be a full lattice in $\mathbb{R}^2$. Then,*

$$|S \cap \Lambda| = 1 + V(S)/\det(\Lambda) + O(r).$$

Proof. See Lemma 2 of [Sch95]. $\square$

Proof of Lemma 29. $\sum_{n \in N_\gamma^*(\alpha,\rho)} \frac{\varphi(n)}{n} \ll \rho N$ follows easily from Lemmas 3.3 and 3.4 of [Cho18], so we will focus on proving $\sum_{n \in N_\gamma^*(\alpha,\rho)} \frac{\varphi(n)}{n} \gg \rho N$. We first claim that there exists a GAP $P := P(b, A_1, A_2, N_1, N_2) \subseteq N_\gamma^*(\alpha, \rho)$. That such a P is contained in $N_\gamma(\alpha, \rho)$ is precisely the content of Lemma 28, and it remains to show that $P \subseteq [\frac{N}{20}, N]$ too. Indeed, $b \geq \frac{N}{20}$, and looking at the maximum element, we have $b + A_1 N_1 + A_2 N_2 \leq \frac{N}{2} + A_1 N_1 + A_2 N_2 \approx \frac{N}{2} + \frac{\rho A_1 A_2}{10} + \frac{N}{10}$. Thus, it suffices to prove the bound $\frac{\rho A_1 A_2}{10} \leq \frac{N}{10}$. Indeed, arguing by cases,

$$\begin{aligned} N < \rho q_{t-1} q_t &\implies A_1 = q_{t-2}, A_2 = q_{t-1} \\ &\implies \rho A_1 A_2 \leq \rho q_{t-1}^2 \leq N \\ &\implies \frac{\rho A_1 A_2}{10} \leq \frac{N}{10}. \end{aligned}$$

Similarly,

$$\begin{aligned} N \geq \rho q_{t-1} q_t &\implies A_1 = q_{t-1}, A_2 = q_t \\ &\implies \rho A_1 A_2 = \rho q_{t-1} q_t \leq N \\ &\implies \frac{\rho A_1 A_2}{10} \leq \frac{N}{10}. \end{aligned}$$

We will sum over this subset P and show that $\sum_{n \in P} \frac{\varphi(n)}{n} \asymp |P| = N_1 N_2$. By the *AM-GM inequality*, one has $\frac{1}{|P|} \sum_{n \in P} \frac{\varphi(n)}{n} \geq \left(\prod_{n \in P} \frac{\varphi(n)}{n}\right)^{\frac{1}{|P|}}$. Thus, defining

$X := \left(\prod_{n\in P} \frac{\varphi(n)}{n}\right)^{\frac{1}{|P|}}$, it suffices to show that $X \gg 1$. Now, $\frac{\varphi(n)}{n} = \prod_{p|n}(1-\frac{1}{p})$, so by interchanging the order of the products and grouping by primes p, we have:

$$\prod_{n\in P} \frac{\varphi(n)}{n} = \prod_{n\in P}\prod_{p|n}\left(1-\frac{1}{p}\right) = \prod_{p\ \text{prime}}\left(1-\frac{1}{p}\right)^{\#\{n\in P: p|n\}}.$$

Define $\alpha_p := \frac{1}{|P|}\cdot \#\{n \in P : p \mid n\}$, the proportion of elements of P divisible by p. Thus, $X = \left(\prod_p (1-\frac{1}{p})^{\alpha_p\cdot|P|}\right)^{\frac{1}{|P|}} = \prod_p (1-\frac{1}{p})^{\alpha_p}$. Our goal is to find an upper bound for α_p. Suppose $\alpha_p > 0$, so that $p \le N$ (Since each $n \in P$ is at most N). This allows us to fix $1 \le n_1^* \le N_1, 1 \le n_2^* \le N_2$, such that $p \mid b + A_1 n_1^* + A_2 n_2^*$. Then, $p \mid n$ for $n = b + A_1 n_1 + A_2 n_2$ *if and only if* $p \mid A_1(n_1 - n_1^*) + A_2(n_2 - n_2^*)$. Set $n_1' := n_1 - n_1^*$ and $n_2' := n_2 - n_2^*$. We must count the number of integer solutions to $A_1 n_1' + A_2 n_2' \equiv 0 \pmod p$, in the box $B := [-N_1, N_1] \times [-N_2, N_2]$. We argue by cases.

If $p \mid A_1$, we have $A_2 n_2' \equiv 0 \pmod p$. Since $\gcd(A_2, p) = 1$, we have $n_2' \equiv 0 \pmod p$. Thus, the number of desired pairs $(n_1', n_2') \in B$ is:

$$(2N_1+1) \times \left\lfloor \frac{2N_2+1}{p} \right\rfloor \le (2N_1+1)\left(\frac{2N_2}{p}+1\right).$$

Thus,

$$|P|\alpha_p \le (2N_1+1)\left(\frac{2N_2}{p}+1\right) \ll N_1\left(1+\frac{N_2}{p}\right) + N_2\left(1+\frac{N_1}{p}\right),$$

which implies

$$\begin{aligned}\alpha_p &\le \frac{N_1}{N_1N_2} + \frac{N_1N_2/p}{N_1N_2} + \frac{N_2}{N_1N_2} + \frac{N_2N_1/p}{N_1N_2}\\ &\ll N_2^{-1} + p^{-1} + N_1^{-1} + p^{-1}\\ &\ll p^{-1} + N_1^{-1} + N_2^{-1}.\end{aligned}$$

Now, by item (4) of Lemma 28, $N_1, N_2 \ge N^\epsilon \ge p^\epsilon \implies N_1^{-1}, N_2^{-1} \le p^{-\epsilon}$. Thus, $\alpha_p \ll p^{-\epsilon}$. The argument holds almost verbatim if we assumed $p \mid A_2$ instead. We will show that this upper bound works for the other case too.

If p divides neither A_1 nor A_2, we will apply Lemma 33. Take $S := [-N_1, N_1] \times [-N_2, N_2]$ as the box of volume $V(S) \approx 4N_1N_2$, and $\Lambda := \{(n_1', n_2') : A_1 n_1' + A_2 n_2' \equiv 0 \pmod p\}$ with fundamental volume $\det(\Lambda) = \frac{p}{\gcd(A_1, A_2)} = p$. Clearly, S is contained in the compact disc of radius $r := \max(N_1, N_2)$. Thus, this gives us:

$$|S \cap \Lambda| = 1 + \frac{4N_1N_2}{p} + O(N_1 + N_2) \implies |P|\alpha_p \ll \frac{N_1N_2}{p} + N_1 + N_2.$$

Therefore,

$$\begin{aligned}\alpha_p &= \frac{N_1N_2}{p|P|} + \frac{N_1}{|P|} + \frac{N_2}{|P|} \\ &\ll \frac{1}{p} + \frac{1}{N_1} + \frac{1}{N_2} \quad (\text{Since } |P| \asymp N_1N_2) \\ &\ll p^{-\epsilon}.\end{aligned}$$

Finally, set $Y = \log\left(\frac{1}{X}\right)$. Using the inequality $\log(1+x) \leq x$ for all $x > -1$,

$$\begin{aligned}Y &= \log \prod_p (1 - \frac{1}{p})^{-\alpha_p} \\ &= -\sum_p \alpha_p \log(1 - \frac{1}{p}) \\ &= \sum_p \alpha_p \log\left(\frac{p}{p-1}\right) \\ &= \sum_p \alpha_p \log\left(1 + \frac{1}{p-1}\right) \\ &\ll \sum_p p^{-\epsilon} \log(1 + \frac{2}{p}) \\ &\ll \sum_p p^{-\epsilon} \cdot \frac{2}{p} \\ &\ll \sum_p p^{-1-\epsilon} \\ &\ll \sum_{n \in \mathbb{Z}^+} \frac{1}{n^{1+\epsilon}} \\ &< \infty.\end{aligned}$$

Thus, $Y \ll 1$, implying that $X \gg 1$.

□

5.1. Completing the proof of the metric Littlewood conjecture

As mentioned earlier, Lemmas 28 and 29 are essential for us to bound the sums $T_N(\alpha, \gamma) := \sum_{n \leq N, n \notin B} \frac{1}{\|n\alpha - \gamma\|}$ and $T_N^*(\alpha, \gamma) := \sum_{n \leq N, n \notin B} \frac{\varphi(n)}{n\|n\alpha - \gamma\|}$. By definition, $T_N \geq T_N^*$. We define the set $B := \{n \in \mathbb{N} : \|n\alpha - \gamma\| < n^{-4\epsilon}\}$.

Proposition 34. *We have $T_N(\alpha, \gamma) \ll N \log N$ and $T_N^*(\alpha, \gamma) \gg N \log N$. Thus,*

$$T_N(\alpha, \gamma) \asymp T_N^*(\alpha, \gamma) \asymp N \log N.$$

Proof. The sharp upper bound for T_N follows from results obtained by Beresnevich-Haynes-Velani; see Proposition 4.1 in [Cho18] for the details. For the sharp lower bound of T_N^*, we restrict n to $[\frac{N}{20}, N]$ and enforce that $\|n\alpha - \gamma\| \geq N^{-2\epsilon}$. Since $n \leq N$, $n^{-4\epsilon} \geq N^{-4\epsilon}$, and for N large, $N^{-2\epsilon} > N^{-4\epsilon}$, so $n \notin B$. Thus, it suffices to prove that the restricted sum

$$\sum_{\frac{N}{20} \leq n \leq N,\ N^{-2\epsilon} \leq \|n\alpha - \gamma\| \leq N^{-\epsilon}} \frac{\varphi(n)}{n\|n\alpha - \gamma\|} \gg N \log N.$$

Partition $[N^{-2\epsilon}, N^{-\epsilon}]$ to smaller, disjoint intervals $(\delta\rho, \rho]$ with $\delta \in (0,1)$. Choose δ small in terms of α. The number of subintervals required to cover this interval is proportional to $\log_{1/\delta}(N^\epsilon) \asymp \epsilon \log N \gg \log N$.

For each subinterval $(\delta\rho, \rho]$, consider the annulus $N_\gamma^*(\alpha, \rho) \setminus N_\gamma^*(\alpha, \delta\rho)$. Then, the denominator $\|n\alpha - \gamma\| \leq \rho$, so by pulling out ρ^{-1} as a lower bound, we obtain

$$\sum_{n \in N_\gamma^*(\alpha,\rho) \setminus N_\gamma^*(\alpha,\delta\rho)} \frac{\varphi(n)}{n\|n\alpha - \gamma\|} \geq \rho^{-1} \left(\sum_{n \in N_\gamma^*(\alpha,\rho)} \frac{\varphi(n)}{n} - \sum_{n \in N_\gamma^*(\alpha,\delta\rho)} \frac{\varphi(n)}{n} \right).$$

By Lemma 29,

$$\sum_{n \in N_\gamma^*(\alpha,\rho)} \frac{\varphi(n)}{n} \asymp \rho N, \quad \sum_{n \in N_\gamma^*(\alpha,\delta\rho)} \frac{\varphi(n)}{n} \asymp \delta\rho N.$$

Thus, this gives us a lower bound of $\rho^{-1}(C_1\rho N - C_2\delta\rho N) = N(C_1 - C_2\delta)$, where C_1 and C_2 are constants. Since δ is small, $C_1 - C_2\delta > 0$. Thus, the sum over a single subinterval is $\gg N$, and since we have $\gg \log N$ subintervals, it follows that the sum is $\gg N \log N$. $\square$

Proof of Proposition 25. Define $\Psi(n) := \frac{\psi(n)}{\|n\alpha - \gamma\|} \mathbf{1}_{n \notin B}$. Define

$$\widetilde{\Psi}(n) := \min\{\Psi(n), 1\}.$$

Suppose $\Psi(n) \geq 1$ for infinitely many n. Since $\|n\beta\| \leq \frac{1}{2} < 1$, we then have $\|n\beta\| < \Psi(n)$ for infinitely many n and every β, so the desired conclusion follows immediately.

Thus, assume that $\Psi(n) < 1$ for all sufficiently large n. Consequently, $\widetilde{\Psi}(n) = \Psi(n)$ for all sufficiently large n. Therefore, replacing Ψ with $\widetilde{\Psi}$ changes only finitely many terms and does not affect divergence or any of the asymptotic estimates below. Moreover,

$$\|n\beta\| < \widetilde{\Psi}(n) \implies \|n\beta\| < \Psi(n).$$

Hence, it is enough to apply Theorem 15 to the bounded approximating sequence $\widetilde{\Psi}$. It suffices to prove

$$\sum_{n=1}^{\infty} \frac{\varphi(n)}{n} \widetilde{\Psi}(n) = \infty \quad \text{and} \quad \sum_{n \leq N} \frac{\varphi(n)}{n} \widetilde{\Psi}(n) \gg \sum_{n \leq N} \widetilde{\Psi}(n).$$

Since $\widetilde{\Psi} = \Psi$ for all but finitely many n, we henceforth make this identification, suppressing finitely many initial terms.

By hypothesis, $\sum_{n=1}^{\infty} \psi(n) \log n = \infty$, so it suffices to prove the estimates

$$\sum_{n \leq N} \frac{\varphi(n)}{n} \Psi(n) \gg \sum_{n \leq N} \psi(n) \log n$$

and

$$\sum_{n \leq N} \Psi(n) \ll 1 + \sum_{n \leq N} \psi(n) \log n.$$

Let N_0 be sufficiently large such that Proposition 34 applies for all $n \geq N_0$. Recall the summation by parts formula: if $A_n := \sum_{m \leq n} a_m$ and b_n is any sequence, then

$$\sum_{n=1}^{N} a_n b_n = A_N b_{N+1} + \sum_{n=1}^{N} A_n (b_n - b_{n+1}).$$

We apply this with $b_n = \psi(n)$. Since ψ is monotone decreasing, we have $\psi(n) - \psi(n+1) \geq 0$. First, set $a_n^* := \frac{\varphi(n)}{n\|n\alpha - \gamma\|} \mathbf{1}_{n \notin B}$, $A_n^* := \sum_{m \leq n} a_m^* = T_n^*(\alpha, \gamma)$. Then,

$$\begin{aligned} \sum_{n \leq N} \frac{\varphi(n)}{n} \Psi(n) &= \sum_{n \leq N} a_n^* \psi(n) \\ &= \psi(N+1) T_N^*(\alpha, \gamma) + \sum_{n=1}^{N} \big(\psi(n) - \psi(n+1)\big) T_n^*(\alpha, \gamma). \end{aligned}$$

By Proposition 34, for $n \geq N_0$, we have $T_n^*(\alpha, \gamma) \gg n \log n$. Therefore,

$$\sum_{n \leq N} \frac{\varphi(n)}{n} \Psi(n) \gg \psi(N+1) N \log N + \sum_{n=N_0}^{N} \big(\psi(n) - \psi(n+1)\big) n \log n.$$

Since $\sum_{m \leq n} \log m \asymp n \log n$, we obtain

$$\sum_{n \leq N} \frac{\varphi(n)}{n} \Psi(n) \gg \psi(N+1) \sum_{m \leq N} \log m + \sum_{n=N_0}^{N} \big(\psi(n) - \psi(n+1)\big) \sum_{m \leq n} \log m.$$

Applying summation by parts in reverse gives

$$\psi(N+1) \sum_{m \leq N} \log m + \sum_{n=N_0}^{N} \big(\psi(n) - \psi(n+1)\big) \sum_{m \leq n} \log m \gg \sum_{n=N_0}^{N} \psi(n) \log n.$$

Hence,

$$\sum_{n\le N}\frac{\varphi(n)}{n}\Psi(n)\gg\sum_{n=N_0}^{N}\psi(n)\log n.$$

Since the finitely many terms with $n<N_0$ are irrelevant for divergence, this proves

$$\sum_{n\le N}\frac{\varphi(n)}{n}\Psi(n)\gg\sum_{n\le N}\psi(n)\log n.$$

Next, set $a_n:=\frac{1}{\|n\alpha-\gamma\|}\mathbf{1}_{n\notin B},\ A_n:=\sum_{m\le n}a_m=T_n(\alpha,\gamma)$. Then,

$$\begin{aligned}\sum_{n\le N}\Psi(n)&=\sum_{n\le N}a_n\psi(n)\\&=\psi(N+1)T_N(\alpha,\gamma)+\sum_{n=1}^{N}\big(\psi(n)-\psi(n+1)\big)T_n(\alpha,\gamma).\end{aligned}$$

For $n\ge N_0$, Proposition 34 gives $T_n(\alpha,\gamma)\ll n\log n$. Therefore,

$$\begin{aligned}\sum_{n\le N}\Psi(n)&\ll 1+\psi(N+1)N\log N+\sum_{n=N_0}^{N}\big(\psi(n)-\psi(n+1)\big)n\log n\\&\ll 1+\psi(N+1)\sum_{m\le N}\log m+\sum_{n=N_0}^{N}\big(\psi(n)-\psi(n+1)\big)\sum_{m\le n}\log m.\end{aligned}$$

Applying summation by parts in reverse again, we get

$$\sum_{n\le N}\Psi(n)\ll 1+\sum_{n=N_0}^{N}\psi(n)\log n\ll 1+\sum_{n\le N}\psi(n)\log n.$$

□

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See Hai Ong
Department of Mathematics, National University of Singapore, Singapore
Email: e1117760@u.nus.edu

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