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Simple matroids and Alfred North Whitehead's theory of dimension (1906)

Thomas Hales

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Abstract

We give a correspondence between simple matroids and a reconstruction of Whitehead's theory of dimension, as found in *On Mathematical Concepts of the Material World* (MW), published in 1906 [8]. In brief, if (E, ϕ) is a geometrical system in the generalized sense of Whitehead, where E is finite and ϕ-maximal, then E is a simple matroid. (*Generalized* here means that Whitehead's three-dimensional axiom has been replaced with finite-dimensionality.) Within the context of the generalized geometrical axioms, *ϕ-maximal* subsets correspond with simple submatroids. Conversely, every simple matroid is a ϕ-maximal geometrical system in the generalized sense of Whitehead.

Matroids were introduced in 1935 by Hassler Whitney [9] and independently by Takeo Nakazawa the same year [5]. As originally conceived by Whitney, matroid theory was a common generalization of the notion of linear independence in vector spaces and circuits in graphs. Nakazawa gave axioms for linear dependence: properties that abstractly describe sequences of vectors $a_1, \ldots, a_s$ whose product $a_1 a_2 \cdots a_s = 0$ is zero in the exterior algebra of a vector space. In 1936, Nakazawa connected his theory explicitly with Whitney's. The history of matroid theory is documented in [3], but the book does not mention MW.

Whitehead's theory is a precursor to their work, although there is no evidence of any direct influence of Whitehead on Whitney or Nakazawa. Some social connections between Whitehead and Whitney's circle of associates at Harvard are mentioned in a remark at the end of this article, but these might amount to no more than a historical curiosity.

According to Whittaker, MW is "extremely difficult reading, the propositions being established by aid of the Peano symbolism: and it fell completely flat so far as enlisting the attention of other writers was concerned" [10]. Indeed, much of MW is written in the unwelcoming symbolic notation of Whitehead and Russell's *Principia Mathematica*. Yet Whitehead held MW in surprisingly high esteem. "Looking backward thirty years later, when he was a famous philosopher at Harvard, he [Whitehead] told me [Lowe] that insofar as he could lay claim to any originality, he thought *'On Mathematical Concepts of the Material World'* was the most original thing he had done" [4, v1 p296]. Whitehead also stated that his theory of dimension was one of the main objects of the memoir.

Familiarity with MW and Oxley's book on matroids is assumed [6]. Whitehead's notation will be used for his theory, and Oxley's notation will be used for matroids. In what follows, the notation O.X refers to numbered results in Oxley, and notation W.X refers to numbered results in Whitehead's MW.

In this article, all matroids are assumed to have a finite ground set. The definition of a matroid can be presented in several ways – in terms of flats, a closure operator, a rank function, or independent sets. All of the definitions turn out to be equivalent, and the equivalence of these definitions (among others such as circuits) is the starting point of matroid theory.

Whitehead's MW has an analogue of each of these fundamental concepts of matroid theory. His name for flats and closure was the ϕ-common region; his name for rank was the ϕ-dimension number; and his name for independent set was ϕ-axial. Both ϕ-maximality and the simple matroid property are hereditary under passage to subsets. Within the context of his geometrical axioms, ϕ-maximal subsets correspond with simple submatroids. A significant part of MW is devoted to proving propositions about these concepts and their relationships.

The analogy between simple matroids and Whitehead's theory is shown in the following table. (The backquote ‛ denotes function application and is a peculiarity of Whitehead's notation that can be disregarded here.)

Simple Matroid	**Whitehead**
closure operator	ϕ-common region of u, or $\mathrm{cm}^{‛}_{\phi} u$
having the same closure	ϕ-equivalent
rank	ϕ-dimension number of u, or $\dim^{‛}_{\phi} u$
independent set	ϕ-axial (or ϕ-prime in a ϕ-maximal subset)
simple matroid	ϕ-maximal (in the presence of other axioms)

Whitehead's theory includes examples that lie outside the scope of simple matroid theory. It is only by imposing extra conditions on Whitehead's theory – that E is finite and ϕ-maximal – that we obtain a correspondence with simple matroid theory. Yet even without these extra conditions, numerous propositions in MW have close analogues in simple matroid theory. Theorems 1 and 2 give the relationship between simple matroids and a reconstruction of Whitehead's theory. We modify Whitehead's original axioms by replacing his three-dimensional axiom (ν) with an axiom of finite-dimensionality (ν'), as follows.

Definition (Whitehead's generalized geometrical axioms)**.** *Let E be a set, and let some of the subsets of E be designated as ϕ-classes. We say that (E, ϕ) is* geometrical (or a geometrical system) in the generalized sense of Whitehead *if the following five axioms hold.*

(λ) *E is a ϕ-class.*

(μ) *For all $x \in E$, the singleton $\{x\}$ is a ϕ-class.*

(ν') *The ϕ-dimension number of E is finite, and $cm^{\backprime}_{\phi}\emptyset = \emptyset$.*

(π) *For all $u \subseteq E$ and for all ϕ-axial subsets v of its ϕ-common region $cm^{\backprime}_{\phi}u$, there exists a subset $w \subseteq E$ such that $v \cup w$ is ϕ-axial and ϕ-equivalent to u.*

(ρ) *For all ϕ-axial subsets $u, v \subseteq E$, if $|u \cap v| \geq 2$, then $u \cup v$ is ϕ-maximal.*

Remark 1. Whitehead defines ϕ to be *geometrical* if it satisfies five axioms $(\lambda, \mu, \nu, \pi, \rho)$_Hp_$\phi$ (see §W.10). The axioms λ, μ, π and ρ are Whitehead's axioms λ_Hp_ϕ, μ_Hp_ϕ, π_Hp_ϕ, and ρ_Hp_ϕ. (Hp_ϕ is Whitehead's abbreviation for *hypothesis on ϕ-classes.*)

Whitehead's axiom ν_Hp_ϕ is that E is three-dimensional. He makes this assumption because his interest is three-dimensional geometry, while observing that "the reasoning can be applied to higher dimensions, only more elaborate inductions and an extra axiom are required." Whitehead does not spell out this extra axiom. We have replaced his axiom with finite-dimensionality (ν'). Whitehead proves that if $|E| \neq 1$, then $\mathrm{cm}^{\backprime}_{\phi}\emptyset = \emptyset$ (see Prop. W.11.12, whose proof assumes $|E| \neq 1$). We have included this identity as part of axiom (ν') to avoid exceptions when E is a singleton. The identity is used in the proof of no loops in the associated matroid. When finite-dimensionality (ν') replaces three-dimensionality (ν_Hp_ϕ), we refer to the geometrical axioms in the *generalized* sense of Whitehead.

The first theorem states that every finite ϕ-maximal geometrical system in the generalized sense of Whitehead is a simple matroid.

Theorem 1. *Let (E, ϕ) be geometrical in the generalized sense of Whitehead. If E is finite and ϕ-maximal, then E is a simple matroid whose closure operator is $u \mapsto cm^{\grave{}}_{\phi} u$. Moreover, a subset of E is a flat of the matroid iff it is an intersection of ϕ-classes. The independent sets of the matroid are the ϕ-axial sets (together with the empty set). Finally, the rank function on the matroid is equal to the ϕ-dimension number on every nonempty set.*

The proof of the theorem relies on new inductive proofs of some propositions that were limited in Whitehead to three dimensions. The proof of the theorem is given below. First, we make a series of remarks.

Remark 2. Two different families of ϕ-classes on E determine the same matroid iff they determine the same collection of flats; that is, they both give the same collection of intersections.

Remark 3. In an elucidatory note, Whitehead indicates that ϕ-classes are intended to generalize the geometrical idea of flatness in Euclidean space (see W.3.12). The use of the term *flat* in both Whitehead and matroid theory points to a shared intuitive understanding of what the theory is about.

Remark 4. In Whitehead's terminology, a nonempty set is ϕ-prime if it does not have a proper subset with the same ϕ-common region. A ϕ-prime set is ϕ-axial if it has the largest cardinality among ϕ-prime sets with the same ϕ-common region. The ϕ-dimension number of a set is defined to be the cardinality of a ϕ-axial set with the same ϕ-common region. Whitehead's theory includes examples such as a pair of skew lines: a set with two elements whose affine closure is three-dimensional; a pair of skew lines is ϕ-prime but not ϕ-axial. In a matroid, there is no distinction between (the analogues of) ϕ-prime and ϕ-axial; that is, all bases of the same flat have the same cardinality. Whitehead's ϕ-maximal condition rules out examples such as skew lines to ensure a match between the ϕ-dimension number and the cardinality of ϕ-prime sets.

Remark 5. Whitehead's definitions can also be applied to infinite sets or even proper classes. If E is finite or infinite, and if every subset of E is declared to be a ϕ-class, then the ϕ-dimension number of every nonempty subset $u \subseteq E$ is equal to its cardinality.

Remark 6. Since Whitehead does not assume that E is finite, simple matroids of finite rank might be a better fit to Whitehead's geometrical axioms than simple matroids on a finite ground set. This is a matter for future investigation. However, we confine our attention to finite ground sets.

Remark 7. To give more historical context, in his earlier *A Treatise on Universal Algebra* (1898), Whitehead gave an exposition of properties of linear dependence and independence, including dependence and independence in the exterior algebra [7, §64,§91–§96]. The propositions include the statement that the maximum number n of independent elements is one more than the dimension of the corresponding projective space (called the "positional manifold") and that any set of n (but no fewer) independent elements span.

Remark 8. Assuming the geometrical axioms, by Prop. W. 12.12, every nonempty subset of E has a set of ϕ-axes. This implies that the ϕ-dimension number is defined (and is a cardinal number) for every nonempty subset of E. (This remark is needed, because Whitehead's general specification of ϕ-dimension skirts the issue of its definedness.)

Remark 9. Our proofs will use Whitehead's propositions, which are consequences of his five axioms of geometry, but it can be checked that whenever we invoke Whitehead's propositions, the proofs go through using only $\lambda, \mu, \nu', \pi, \rho$. In fact, we will never need to use the axiom ρ. (However, Theorem 1 assumes that E is ϕ-maximal, which is a strong form of axiom ρ.) If the proposition number is W.X with $X < 10$, then the proposition occurs before the statement of the axioms in section 10 of MW, and there is nothing to check. The propositions in the early sections of Whitehead include results that correspond in matroid theory with the basic properties of the closure operator of matroids, the hereditary property of independent sets, the relation between the independent sets and closure, and the simplicity property.

We will repeatedly use the following two propositions from Whitehead, which give relations between the ϕ-dimension number and ϕ-common region.

Proposition. *W.12.21.* *Let (E, ϕ) be a geometrical system in the generalized sense of Whitehead. Assume that $u \subseteq E$, $\varnothing \neq v \subseteq E$, and $cm^{\backprime}_{\phi} v \subseteq cm^{\backprime}_{\phi} u$. Then* $\dim^{\backprime}_{\phi} v \leq \dim^{\backprime}_{\phi} u$.

Proof. See MW. □

In particular, $\dim^{\backprime}_{\phi} E < \infty$ is an upper bound on the ϕ-dimension number of nonempty subsets of E.

Proposition. *W.12.23.* *Let (E, ϕ) be a geometrical system in the generalized sense of Whitehead. Assume that $u \subseteq E$, $\varnothing \neq v \subseteq E$, and $cm^{\backprime}_{\phi} v \subseteq cm^{\backprime}_{\phi} u$. Then*

$$cm^{\backprime}_{\phi} v = cm^{\backprime}_{\phi} u \quad \textit{iff} \quad \dim^{\backprime}_{\phi} v = \dim^{\backprime}_{\phi} u.$$

Proof. See MW. □

Before proving the theorem, we give two key propositions (Props. W.12.37 and W.13.11) that appear in MW whose proofs need to be updated to our assumption context. As mentioned above, Whitehead observed "the reasoning can be applied to higher dimensions, only more elaborate inductions ... are required." Here we supply the more elaborate inductions. We add Lemma 1, which is not found in Whitehead, that will be used three times to update Whitehead's proofs. The lemma serves as an induction step in other proofs. Prop. W.12.37 gives the existence of a ϕ-prime subset of a given nonempty set. Prop. W.13.11 is the hereditary property of ϕ-maximal subsets. It will be used to show in the proof of Theorem 1 that the ϕ-prime sets are in fact ϕ-axial.

Proposition. *W.12.37.* *Let (E, ϕ) be a geometrical system in the generalized sense of Whitehead. Let $\emptyset \neq u \subseteq E$. Then there exists a (finite) ϕ-prime set $u' \subseteq u$ that is ϕ-equivalent to u.*

Proof. Assume $\emptyset \neq u \subseteq E$. We claim that if $v \subseteq u$ is any finite subset of u of maximal dimension among finite subsets of u, then v is ϕ-equivalent to u. We prove the contrapositive of the claim. Assume that v is a finite subset of u that is not ϕ-equivalent to u. Pick $x \in u \setminus \mathrm{cm}^{\backslash}_{\phi} v$ (using Props. W.4.21, W.4.25). Then $\mathrm{cm}^{\backslash}_{\phi} v \subsetneq \mathrm{cm}^{\backslash}_{\phi}(v \cup \{x\})$ and $\dim^{\backslash}_{\phi} v < \dim^{\backslash}_{\phi}(v \cup \{x\}) \leq \dim^{\backslash}_{\phi} u$ (by Props. W.12.21, W.12.23). This shows that v does not have maximal dimension among finite subsets of u. This proves the claim.

Let v be a finite subset of u that is ϕ-equivalent to u. Let $u' \subseteq v$ be a subset of v of minimal cardinality among those subsets of v that are ϕ-equivalent to u. Then u' is clearly ϕ-prime and ϕ-equivalent to u. □

Lemma 1. *Let (E, ϕ) be a geometrical system in the generalized sense of Whitehead. Let $v \subseteq E$ be ϕ-prime, and let $x \in E$ be such that $v \cup \{x\}$ is ϕ-maximal and such that $x \notin \mathrm{cm}^{\backslash}_{\phi} v$. Then v and $v \cup \{x\}$ are ϕ-axial. Also $\dim^{\backslash}_{\phi}(v \cup \{x\}) = \dim^{\backslash}_{\phi} v + 1$.*

Proof. Choose $v' \subseteq v \cup \{x\}$ that is ϕ-equivalent to $v \cup \{x\}$ and ϕ-prime. By the definition of ϕ-maximal, we have that v' is ϕ-axial. Also, v' is finite. We have

$$|v| \leq \dim^{\backslash}_{\phi} v < \dim^{\backslash}_{\phi}(v \cup \{x\}) = \dim^{\backslash}_{\phi} v' = |v'| \leq |v| + 1.$$

All weak inequalities must be equalities. This implies that $|v| = \dim^{\backslash}_{\phi} v$, that $v' = v \cup \{x\}$, and that $|v| + 1 = |v \cup \{x\}| = \dim^{\backslash}_{\phi}(v \cup \{x\})$. The conclusion follows. □

Proposition. *W.13.11.* *Let (E, ϕ) be a geometrical system in the generalized sense of Whitehead. Every nonempty subset of a ϕ-maximal set is also ϕ-maximal.*

Proof. Assume u is ϕ-maximal. We argue by contradiction, and assume that u has a nonempty subset that is not ϕ-maximal. Among all such nonempty subsets of u we choose a subset u' whose ϕ-dimension number is as large as possible. We have $\operatorname{dim}^{\backslash}_{\phi} u' < \operatorname{dim}^{\backslash}_{\phi} u$ (by Props. W.12.21, W.12.23, W.4.21). Let v be any subset of u' that is ϕ-prime and ϕ-equivalent to u' (which exists by W.12.37). Let $x \in u \setminus \operatorname{cm}^{\backslash}_{\phi} v$ (this exists). Then $\operatorname{dim}^{\backslash}_{\phi} v < \operatorname{dim}^{\backslash}_{\phi}(v \cup \{x\})$, and by the dimension maximizing choice of u' (and v), we have that $v \cup \{x\}$ is ϕ-maximal. By Lemma 1, we have that v is ϕ-axial. This shows that u' is ϕ-maximal, which is contrary to the choice of u'. □

Proof of (Theorem 1). Let (E, ϕ) be geometrical in the generalized sense of Whitehead. Assume that E is finite and ϕ-maximal. It follows from Prop. W.13.11 that every nonempty subset of E is ϕ-maximal. In particular, a subset of E is ϕ-prime iff it is ϕ-axial. We refer to this particular fact as property (τ).

Let $\mathcal{I}$ be the collection of ϕ-prime (or ϕ-axial) subsets of E together with the empty set. To show that $(E, \mathcal{I})$ is a matroid, we verify the three axioms §O.1.1 of independent sets.

I1 $\emptyset \in \mathcal{I}$: This is true by the choice of $\mathcal{I}$.

I2 We claim that if $I \in \mathcal{I}$ and $I' \subseteq I$, then $I' \in \mathcal{I}$. This is Whitehead's Proposition W.5.23. (Whitehead calls Prop. W.5.23 and two closure laws "the foundation of the whole theory. It is remarkable that it requires no axiom concerning ϕ.")

I3 Assume that I_1 and I_2 are in $\mathcal{I}$ and $|I_1| < |I_2|$. We claim that there exists $x \in I_2 \setminus I_1$ such that $I_1 \cup \{x\} \in \mathcal{I}$. The case $I_1 = \emptyset$ being trivial (by Prop. W.5.233), we assume that $I_1 \neq \emptyset$. *We claim that I_2 is not a subset of $\operatorname{cm}^{\backslash}_{\phi} I_1$.* Otherwise, for a contradiction, we have $\operatorname{cm}^{\backslash}_{\phi} I_2 \subseteq \operatorname{cm}^{\backslash}_{\phi} I_1$ (by Prop. W.4.21, W.4.31), and
$$|I_2| = \operatorname{dim}^{\backslash}_{\phi} I_2 \leq \operatorname{dim}^{\backslash}_{\phi} I_1 = |I_1|$$
(by property τ and W.12.21), which is contrary to assumption. Thus, by the claim, there exists $x \in I_2 \setminus \operatorname{cm}^{\backslash}_{\phi} I_1$. By W.4.25, we have $x \in I_2 \setminus I_1$. Recall from above that $I_1 \cup \{x\}$ is ϕ-maximal. Apply Lemma 1 with $v = I_1$. Hence $I_1 \cup \{x\}$ is ϕ-prime. By definition $I_1 \cup \{x\} \in \mathcal{I}$. This completes the proof of the third axiom of matroids.

This shows that $(E, \mathcal{I})$ is a matroid.

Furthermore, the matroid $(E, \mathcal{I})$ has no loops by Prop. W.5.233. Furthermore, for all $x, y \in E$, we have $\{x, y\} \in \mathcal{I}$ by Prop. W.5.235 and property (τ). Hence, by the definition of simplicity (Oxley, p12), the matroid is simple.

Next we show that the function $u \mapsto \mathrm{cm}^{\backslash}_{\phi} u$ is the closure operator associated with the simple matroid $(E, \mathcal{I})$. To accomplish this, we show that cm_{ϕ} satisfies the characteristic properties of a closure operator in Lemma O.1.4.3 and Theorem O.1.4.5.

CL0 $u \in \mathcal{I}$ iff (for all $x \in u$, $x \notin \mathrm{cm}^{\backslash}_{\phi}(u \setminus \{x\})$); by Prop. W.5.231.

CL1 If $u \subseteq E$, then $u \subseteq \mathrm{cm}^{\backslash}_{\phi} u$ by Prop. W.4.25.

CL2 If $u \subseteq v \subseteq E$, then $\mathrm{cm}^{\backslash}_{\phi} u \subseteq \mathrm{cm}^{\backslash}_{\phi} v$ by Prop. W.4.21.

CL3 If $u \subseteq E$, then $\mathrm{cm}^{\backslash}_{\phi} \mathrm{cm}^{\backslash}_{\phi} u = \mathrm{cm}^{\backslash}_{\phi} u$ by Prop. W.4.31.

CL4 Assume $u \subseteq E$ and $x \in E$ and $y \in \mathrm{cm}^{\backslash}_{\phi}(u \cup \{x\}) \setminus \mathrm{cm}^{\backslash}_{\phi} u$. We claim $x \in \mathrm{cm}^{\backslash}_{\phi}(u \cup \{y\})$. Dismissing a trivial case, we may assume that u is nonempty. By Prop. W.12.37, we may choose a ϕ-prime set $I_1 \subseteq u$ that is ϕ-equivalent to u. Then by Props. W.4.43, W.4.21, we have $y \in \mathrm{cm}^{\backslash}_{\phi}(I_1 \cup \{x\}) \setminus \mathrm{cm}^{\backslash}_{\phi} I_1$. We have $x \notin \mathrm{cm}^{\backslash}_{\phi} I_1$. By Lemma 1, $I_1 \cup \{x\}$ is ϕ-axial, and $\dim^{\backslash}_{\phi}(I_1 \cup \{x\}) = \dim^{\backslash}_{\phi} I_1 + 1$. We have

$$\mathrm{cm}^{\backslash}_{\phi} I_1 \subsetneq \mathrm{cm}^{\backslash}_{\phi}(I_1 \cup \{y\}) \subseteq \mathrm{cm}^{\backslash}_{\phi}(I_1 \cup \{x\}).$$

Comparing dimensions (using Prop. W.12.21 and W.12.23), we find that $I_1 \cup \{y\}$ and $I_1 \cup \{x\}$ have the same ϕ-dimension number and are therefore ϕ-equivalent. This gives

$$x \in \mathrm{cm}^{\backslash}_{\phi}(I_1 \cup \{x\}) = \mathrm{cm}^{\backslash}_{\phi}(I_1 \cup \{y\}) \subseteq \mathrm{cm}^{\backslash}_{\phi}(u \cup \{y\}).$$

This completes the proof of CL4.

Hence cm_{ϕ} is the closure operator cl of the matroid.

The flats of the matroid are those of the form $\mathrm{cm}^{\backslash}_{\phi} u$. By Whitehead's definition of the ϕ-common region, a set is a ϕ-common region iff it is an intersection of ϕ-classes.

Finally, we show that the ϕ-dimension number is the rank of the matroid on nonempty sets. (In Whitehead, the ϕ-dimension number of the empty set is undefined.) Let u be nonempty. Both the rank r and the ϕ-dimension number satisfy the closure property:

$$r(u) = r(\mathrm{cl}(u)), \quad \dim^{\backslash}_{\phi}(u) = \dim^{\backslash}_{\phi}(\mathrm{cl}(u)),$$

by Lemma O.1.4.2 and by the definitions of ϕ-dimension number and ϕ-equivalence. Thus, it is enough to check that the rank is equal to the ϕ-dimension number when $u = I \in \mathcal{I}$ is ϕ-prime and hence also ϕ-axial. In this case, I is an independent set, and

$$r(I) = |I| = \dim^{\backslash}_{\phi} I,$$

by Prop. O.1.3.5 and by the definition of ϕ-axial. This proves that the rank is equal to the ϕ-dimension number. $\square$

The converse theorem states that every simple matroid is a finite ϕ-maximal geometrical system in the generalized sense of Whitehead. To be explicit, the converse theorem chooses the ϕ-classes to be flats, restricts to finite simple matroids, replaces Whitehead's dimension axiom with finite-dimensionality (ν'), and avoids some comparisons of the theories on the empty set.

Theorem 2. *Let $(E, \mathcal{I})$ be a simple matroid, with finite ground set E and independent sets $\mathcal{I}$. Define the ϕ-classes to be the flats (i.e. closed sets) of the matroid. Then*

1. *For any subset u of the ground set E, the ϕ-common region of u is the matroid closure of u.*
2. *Two subsets u, v of the ground set E are ϕ-equivalent iff they have the same matroid closure.*
3. *A nonempty subset of the ground set E is ϕ-prime iff it is an independent set of the matroid.*
4. *The ϕ-dimension number of a nonempty subset of the ground set E is its matroid rank.*
5. *If E is nonempty, then E is ϕ-maximal. Moreover, every ϕ-prime set is ϕ-axial.*
6. *(E, ϕ) is geometrical in the generalized sense of Whitehead.*

Remark 10. Statement (5) shows why it is necessary to introduce the assumption that E is ϕ-maximal in Theorem 1. Statement (6) is the main conclusion of the converse theorem, showing that every simple matroid satisfies the generalized axioms of Whitehead.

Proof. 1. Let u be a subset of the ground set E. By (Oxley, p31, Ex.1c), $\mathrm{cl}(u)$ is the intersection of flats containing u; and hence by definition, $\mathrm{cl}(u) = \mathrm{cm}^{\backprime}_{\phi} u$.

2. This statement follows directly from the definition of ϕ-equivalence and statement (1).

3. Compare Theorem O.1.4.5 with Proposition W.5.231, which give the same criterion for an independent set (resp. nonempty ϕ-prime) in terms of matroid closure (resp. ϕ-common region). Compare CL0 above.

4. Let u be a nonempty subset of the ground set. Since the rank and the ϕ-dimension number both satisfy (O.1.4.2)

$$r(u) = r(\mathrm{cl}(u)), \qquad \dim^{\backprime}_{\phi} u = \dim^{\backprime}_{\phi}(\mathrm{cl}(u)),$$

we may assume without loss of generality that u is a flat. Let $\mathcal{P}$ be the set of subsets I of u that are ϕ-prime and ϕ-equivalent to u. We have $I \in \mathcal{P}$ iff I is an independent set whose closure is u. The set $\mathcal{P}$ is nonempty (i.e. a basis of u exists by O.1.4.10), and all members of $\mathcal{P}$ have the same cardinality. If $I \in \mathcal{P}$,

$$r(u) = r(\mathrm{cl}(I)) = r(I) = |I| = \max_{I \in \mathcal{P}} |I| = \dim^{\backprime}_{\phi} u.$$

5. We show that every ϕ-prime set is ϕ-axial. Let I be ϕ-prime. Then

$$|I| = r(I) = \dim^{\backprime}_{\phi}(I).$$

Hence I is ϕ-axial. This implies that every nonempty subset of E is ϕ-maximal.

6.λ. E is closed by CL1 of O.1.4.3.

6.μ and 6.ν'. Every finite matroid clearly has finite dimension. In a simple matroid, singletons and the empty set are closed, by the circuit test of Prop. O.1.4.11.ii.

6.π. Let $u \subseteq E$ be closed (without loss of generality), and let I be a nonempty independent set contained in u. Let J be a basis of u containing I. The closure of J is u, and J is ϕ-equivalent to u.

6.ρ. From (5), every nonempty subset of E is ϕ-maximal. $\square$

Remark 11. As a historical aside, we mention some significant social connections between Whitehead and Whitney's circle of associates, as mathematicians in the same intellectual milieu. Of course, these social ties have no bearing on the validity of the theorems proved above. Both of them were at Harvard: Whitney as a young PhD (received in 1932), and Whitehead from 1924 until his retirement in 1937. Whitney's PhD advisor was George Birkhoff, who had numerous interactions with Whitehead at Harvard. In fact, the Whiteheads lived two floors above the Birkhoffs on Memorial Drive, and they "were very congenial with them" [2]. Harvard's Society of Fellows grew out of a conversation between Whitehead and the biochemist L. J. Henderson (who, years earlier, had helped to recruit Whitehead to Harvard) [4, v2 p254]. When the Society was organized in 1933, George Birkhoff's son Garrett became one of the inaugural Junior Fellows, and Whitehead became an inaugural Senior Fellow. Garrett Birkhoff was an early contributor to matroid theory, publishing the same year as Whitney in 1935 [1]. Incidentally, another of the six inaugural Junior Fellows was Quine, who received his PhD under Whitehead at Harvard in 1932. The Society of Fellows met for dinner Monday nights. It is noteworthy but possibly merely coincidental that Whitehead and Garrett Birkhoff were in regular contact at dinners during the period when Birkhoff was writing on matroids.

In summary and conclusion, Alfred North Whitehead introduced a theory of dimension in 1906 that is a precursor to the theory of simple matroids. By generalizing his theory to finite dimensions and restricting his theory to finite ϕ-maximal sets, a multi-faceted correspondence is obtained with simple matroid theory. That correspondence connects notions in Whitehead's theory with notions in matroid theory such as rank, independent sets, and closure.

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Thomas Hales
Department of Mathematics, University of Pittsburgh, Pittsburgh, PA
Email: hales@pitt.edu